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THE FUNDAMENTAL DOMAIN OF THE TREE OF $GL(2)$ OVER THE FUNCTION FIELD OF AN ELLIPTIC CURVE

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1. Introduction. Let E be an elliptic curve over a field k defined by a Weierstrass equation $F(x, y) = 0$ where

$$F(x, y) = y^2 + a_1xy + a_3y - x^3 - a_2x^2 - a_4x - a_6.$$

Here k is any field. In particular, k is not assumed to be finite. Let $k[E]$ be its affine coordinate ring, and let $t = x/y$ be a local uniformizer at ∞ (the point at infinity). Then, $k[E]$ can be embedded into $k((t))$ in such a way that $\mathrm{ord}(x) = -2$ and $\mathrm{ord}(y) = -3$, where ord is the order function of $k((t))$. Let $k_\infty = k((t))$ and $\mathcal{O}_\infty = k[[t]]$. We will identify $k[E]$ with its embedding into k_∞. Furthermore, let $\Gamma = GL(2, k[E])$, $K = GL(2, \mathcal{O}_\infty)$, $G = GL(2, k_\infty)$, and Z be the center of G. It is well known that we can define a tree structure $\mathcal{T}$ on G/KZ (see Serre [3] or Section 2 of this paper). Each vertex of $\mathcal{T}$ has exactly $|k| + 1$ vertices adjacent to it. ($|k|$ denotes the cardinality of k.) $\mathcal{T}$ looks like Figure 1 when $k = F_3$ (the field of three elements). Moreover, the quotient graph $\Gamma\backslash\mathcal{T}$ is well defined. The aim of this paper is to determine the shape of $\Gamma\backslash\mathcal{T}$. More specifically, we will define a subtree $\mathcal{S}$ of $\mathcal{T}$ such that $\mathcal{S} \simeq \Gamma\backslash\mathcal{T}$. Thus $\Gamma\backslash\mathcal{T}$ is a tree and $\mathcal{S}$ is a fundamental domain of $\mathcal{T}$ modulo Γ.

To describe the shape of $\mathcal{S}$, we need to consider the k-rational points of E. However, since we do not have to consider E over any extension of k, in the rest of the paper, a rational point of E or a rational solution of $F(x, y) = 0$ always means a k-rational point or a k-rational solution. Moreover, the same letter E is used to denote the set of the rational points of E. Now, the shape of $\mathcal{S}$ (or $\Gamma\backslash\mathcal{T}$) can be informally described as follows.

(1) There is a special vertex called o (which stands for the origin).

(2) For each l in $k \cup \{\infty\}$, there is a vertex $v(l)$ adjacent to o. $v(l)$'s are all different. Thus, there are exactly $|k| + 1$ vertices adjacent to o.

(3) In order to describe the rest of $\mathcal{S}$, let $\mathcal{S}(l)$ be the connected component (subtree) of $\mathcal{S} - \{o\}$ which contains $v(l)$. Thus, $\mathcal{S}$ consists of o and the union of $\mathcal{S}(l)$'s (which are all disjoint for different l). The description of $\mathcal{S}(l)$ is divided into three cases depending on l as follows.

(3.1) Suppose $F(x, y) = 0$ has no rational solution such that $x = l$. In this case, $\mathcal{S}(l)$ consists of only $v(l)$; that is, there is no other vertex adjacent to $v(l)$ except for o. $\mathcal{S}(l)$ together with o is shown in Figure 2.

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