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DISTRIBUTION OF THE ERROR TERM IN THE WEYL ASYMPTOTICS FOR THE LAPLACE OPERATOR ON A TWO-DIMENSIONAL TORUS AND RELATED LATTICE PROBLEMS

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1. Introduction. Let E_n be the eigenvalues of the Laplace-Beltrami operator on a d-dimensional compact smooth closed Riemannian manifold M, $-\Delta\varphi_n = E_n\varphi_n$ and let $N(E) = \#\{E_n \leqslant E\}$ be the spectral function of this operator. The Weyl formula gives the asymptotics of $N(E)$, when $E \to \infty$:

$$N(E) = \frac{\mathrm{Vol}\, M \cdot \mathrm{Vol}\, \Omega_d}{(2\pi)^d} E^{d/2}(1 + o(1)), \tag{1.1}$$

where $\Omega_d = \{x \in \mathbf{R}^d, |x| \leqslant 1\}$. A classical problem is: What is the asymptotic behavior of the error term in the Weyl formula

$$D(E) = N(E) - \frac{\mathrm{Vol}\, M \cdot \mathrm{Vol}\, \Omega_d}{(2\pi)^d} E^{d/2}, \tag{1.2}$$

when $E \to \infty$?

In the present work we study this problem in a simple case, when M is a two-dimensional torus $\mathbf{T}^2(2\pi a_1, 2\pi a_2) = \mathbf{R}^2/(2\pi a_1 \mathbf{Z} \oplus 2\pi a_2 \mathbf{Z})$. In this case, $-\Delta$ is a Laplace operator with periodic boundary conditions, in a rectangle with the sides $2\pi a_1$, $2\pi a_2$. The eigenfunctions of $-\Delta$ are

$$\varphi_n(x) = \exp(i(n_1 x_1/a_1 + n_2 x_2/a_2)),$$

where $n = (n_1, n_2) \in \mathbf{Z}^2$, and the eigenvalues are

$$E_n = n_1^2/a_1^2 + n_2^2/a_2^2;$$

hence

$$N(E) = \#\{n | n_1^2/a_1^2 + n_2^2/a_2^2 \leqslant E\}.$$

So we come to the classical lattice problem: What is the asymptotic behavior, when $E \to \infty$, of the difference $N(E) - S(E)$, between the number of lattice points inside

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