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ON THE RELATION BETWEEN CANTOR'S CAPACITY AND THE SECTIONAL CAPACITY

ROBERT RUMELY

0. Introduction. Recently, Chinburg ([2]) introduced a new kind of capacity for algebraic varieties, which he called the sectional capacity. Earlier, Cantor and Rumely ([1], [5]) had developed another theory of capacity for algebraic curves. This paper establishes the relationship between them: for curves, a single underlying quadratic form governs both.

Let K be a global field and let $\mathscr{V}/K$ be a smooth, projective, geometrically connected variety of dimension d. Chinburg's sectional capacity $S_\gamma(\mathbb{E}, D)$ is a function of an adelic set $\mathbb{E}$ and an ample K-rational divisor D on $\mathscr{V}$ with real coefficients. More precisely, for each place v of K, let K_v be the completion of K at v and let $\tilde{K}_v$ be an algebraic closure of K_v. Then $\mathbb{E} = \prod_v E_v$ where $E_v \subset \mathscr{V}(\tilde{K}_v)$ for each v, and $D = \sum_{i=1}^m r_i X_i$ where $r_i \in \mathbb{R}$ for each i and each X_i is a prime divisor rational over K. Each E_v is assumed to be stable under $\mathrm{Gal}(\tilde{K}_v/K_v)$ and separated from $\mathrm{supp}(D)$ (regarded as a subset of $\mathscr{V}(\tilde{K}_v)$) in the v-adic topology.

Loosely speaking, $S_\gamma(\mathbb{E}, D)$ is the limit, as $M \to \infty$, of the reciprocal of the volume of the set of adelic sections of the line bundle $\mathscr{L}(MD)$ having sup norm $\|f\|_{\mathbb{E}} \leqslant 1$; to make the limit behave well, the volume is actually raised to the power $-(d+1)!/M^{d+1}$. Even if the limit fails to exist, one at least has a lim inf, $S_\gamma^-(\mathbb{E}, D)$, and a lim sup, $S_\gamma^+(\mathbb{E}, D)$. Chinburg showed that when $\mathscr{V} = \mathbb{P}^1/\mathbb{Q}$, $D = (\infty)$, and $\mathbb{E} = E \times \prod_p \tilde{\mathcal{O}}_p$ where $E \subset \mathbb{C}$ and $\tilde{\mathcal{O}}_p$ is the ring of integers of $\tilde{\mathbb{Q}}_p$, then $S_\gamma(\mathbb{E}, (\infty))$ exists and coincides with the classical logarithmic capacity $\gamma_\infty(E)$.

In studying $S_\gamma(\mathbb{E}, D)$ it is natural to fix $\mathbb{E}$ and let D vary over divisors with common support. Given a divisor $D = \sum_{i=1}^m r_i x_i$ with real coefficients, put $\lfloor D \rfloor = \sum_{i=1}^m \lfloor r_i \rfloor X_i$, where $\lfloor r \rfloor$ is the greatest integer function. We will say that D is ample if $\lfloor mD \rfloor$ is very ample, for some integer $m > 0$. Chinburg found that in several examples, as D varied over the ample cone, $\log(S_\gamma(\mathbb{E}, D))$ was given by a homogeneous form of degree $d+1$ in the variables $r_1, \ldots, r_m$.

When $\mathscr{V} = \mathscr{C}$ is a curve, given a set $\mathbb{E} \subset \prod_v \mathscr{C}(\tilde{K}_v)$ and a finite, Galois-stable set of point $\mathscr{X} = \{x_1, \ldots, x_m\}$ in $\mathscr{C}(\tilde{K})$, a natural quadratic form arises in the Cantor-Rumely theory: the "Green's matrix" $\Gamma(\mathbb{E}, \mathscr{X})$, which is a symmetric $m \times m$ real-valued matrix with nonnegative off-diagonal entries, attached to $\mathbb{E}$ and to the support $\mathscr{X} = \{x_1, \ldots, x_m\}$ of the divisors being considered. The matrix $\Gamma(\mathbb{E}, \mathscr{X})$ is defined provided $\mathbb{E}$ and $\mathscr{X}$ satisfy certain mild hypotheses stated in §1. In this paper, we will prove the following theorem, asked by Chinburg:

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