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RIEMANN SURFACES AND ABELIAN VARIETIES WITH AN AUTOMORPHISM OF PRIME ORDER

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1. Introduction. Let M be a compact Riemann surface and $J(M)$ its Jacobi variety obtained from the surface via the matrix of periods of abelian differentials.

A very puzzling problem is that of the determination of the surfaces which admit a maximal group of automorphisms, maximal in the sense that the surfaces have zero moduli. The interest arises on the one hand because one may hope for an explicit computation of $J(M)$ in these cases and on the other hand because they give points where the moduli space can be singular.

The surfaces with a cyclic group of automorphisms are of special significance and were studied by Scorza, Lefschetz, and Weil among others, all of whom obtained partial results. The purpose of this paper is to present the complete solution to this problem. Before going into the details however, we will give a brief sketch of its history.

Scorza [3] considered the surface given by the algebraic equation

$$y^5 = x(x-1)$$

which admits a cyclic group of order 5 of automorphisms—the hyperelliptic involution is unimportant—and asked the question: Why is this surface unique? Though one could argue in terms of Weierstrass points, the question asks for an answer in terms of linear algebra: Why is there only one Jacobi variety admitting a cyclic group of order 5 in genus two? Scorza then went on to compute the Riemann matrix of the curve and all Riemann matrices admitting a cyclic group of order 5. He classified them and showed that they are all conjugate under symplectic matrices; he was, however, unable to generalize his answer even to cyclic groups of orders 7 or 11 in genus 3 and 5, respectively.

In working out the theory of abelian varieties [1], Lefschetz considered the Riemann surfaces of genus $g = (p-1)/2$ given by algebraic equations of the form $y^p = x^a(x-1)$ where a is an integer and where p is a prime, and asked whether for different values of a these surfaces were conformally distinct. It is therefore natural to ask if such a surface admits automorphisms of order p other than the obvious ones. Lefschetz classified the full groups of automorphisms of these surfaces:

if $a = 1$, a group of order $2p$;

if $p = 7$ and $a = 2$, a group of order 168;

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