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MOSTOW RIGIDITY AND THE BISHOP-STEGER DICHOTOMY FOR SURFACES OF VARIABLE NEGATIVE CURVATURE

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1. Introduction. Let M be a compact, orientable, C^∞ surface of genus $\geqslant 2$ and with fundamental group $\Gamma = \pi_1(M)$. Let I_1 and I_2 be isomorphisms of Γ into $PSL(2, \mathbb{R})$, the group of linear fractional transformations $z \to (az + b)/(cz + d)$ where $a, b, c, d \in \mathbb{R}$ and $ad - bc = 1$, which may also be viewed as the group of isometries of the hyperbolic plane H. When are I_1 and I_2 geometrically conjugate; i.e., when does there exist $\varphi \in PSL(2, \mathbb{R})$ such that $\varphi \circ I_1(\gamma) = I_2(\gamma) \circ \varphi$ for all $\gamma \in \Gamma$? Equivalently, when is there an isometry Φ: $H/I_1(\Gamma) \to H/I_2(\Gamma)$ which induces the trivial isomorphism $I_2 \circ I_1^{-1}$ of fundamental groups?

We shall discuss two criteria, the first due to Mostow [Mw], the second to Bishop and Steger [BS]. Mostow's criterion uses the Dehn-Nielsen boundary correspondence ([De], appendix). This is a homeomorphism ψ: $\hat{\mathbb{R}} \to \hat{\mathbb{R}}$ such that $\psi \circ I_1(\gamma) = I_2(\gamma) \circ \psi$ for every $\gamma \in \Gamma$; it is uniquely determined by I_1 and I_2. (Note: $\hat{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ is a circle here.) Mostow's theorem states that I_1 and I_2 are geometrically conjugate if and only if ψ is absolutely continuous, in which case ψ is the restriction to $\hat{\mathbb{R}}$ of a linear fractional transformation $\varphi \in PSL(2, \mathbb{R})$. The Bishop-Steger criterion is based more directly on the geometric actions of $I_1(\Gamma)$ and $I_2(\Gamma)$ on H, equipped with the Poincaré distance d. It states that I_1 and I_2 are geometrically conjugate if and only if, for any (every) $z \in H$ and $0 < s < 1$,

$$\sum_{\gamma \in \Gamma} \exp\{-sd(I_1(\gamma)z, z) - (1 - s)d(I_2(\gamma)z, z)\} = \infty.$$

Moreover, if I_1 and I_2 are not geometrically conjugate, then there exists $\delta > 0$ (depending on s but not z) such that

$$\sum_{\gamma \in \Gamma} \exp\{(1 - \delta)\{-sd(I_1(\gamma)z, z) - (1 - s)d(I_2(\gamma)z, z)\}\} < \infty.$$

The purpose of this paper is to prove analogous theorems for surfaces of variable negative curvature and to exhibit their close connection with the ergodic theory of the associated geodesic flows. Let g_1 and g_2 be C^∞ Riemannian metrics on M, each with strictly negative curvature at every point of M. Assume that the geodesic flows associated with g_1 and g_2 both have topological entropy 1. (Note: Multiplying a

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