Vol. 67, No. 3 DUKE MATHEMATICAL JOURNAL © September 1992

A NECESSARY CONDITION FOR THE EXISTENCE OF COMPACT CLIFFORD-KLEIN FORMS OF HOMOGENEOUS SPACES OF REDUCTIVE TYPE

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1. Introduction and statement of results. A *compact Clifford-Klein form* of a homogeneous space G/H is the quotient $\Gamma\backslash G/H$ if Γ is a subgroup of G acting properly discontinuously and freely on G/H so that $\Gamma\backslash G/H$ is compact in the quotient topology. Our concern here is with the following problem.

PROBLEM 1.1. *Find a criterion for the existence of compact Clifford-Klein forms of a homogeneous space G/H.*

In this paper we treat the case where G/H is of reductive type (see §2 for definition). Semisimple orbits in $\mathfrak{g} = \mathrm{Lie}(G)$ and semisimple symmetric spaces are typical examples. A distinguishing feature in our setting is that H is noncompact, and consequently the action of a discrete subgroup of G on G/H is not automatically properly discontinuous. So far, the following facts on discontinuous groups have been known.

FACT 1.2 (see §2 for notation). *Let G/H be a homogeneous space of reductive type.*

(1) (Calabi-Markus phenomenon; [CM; Wo1; Wo2; Ku; Ko1]) *Only finite subgroups of G can act properly discontinuously on G/H if and only if* $\mathbb{R}$-rank $G =$ $\mathbb{R}$-rank H.

(2) (A necessary condition; [Ko1] Proposition 4.10) *If* rank $G =$ rank H *and if* c-rank $G > c$-rank H, *then G/H does not have a compact Clifford-Klein form.*

(3) (A sufficient condition; [BoHC;Bo;Ku;Ko1]) *If there exists a subgroup G' reductive in G such that* $\mathfrak{a}(G') \cap W_G \cdot \mathfrak{a}(H) = \{0\}$ *and* $\mathrm{d}(G') + \mathrm{d}(H) = \mathrm{d}(G)$, *then G/H has a compact Clifford-Klein form.*

An existence result of compact Clifford-Klein forms (or noncompact ones of finite volume) of Riemannian symmetric spaces [BoHC; Bo] has opened a theory of Eisenstein series in harmonic analysis on square integrable functions over the double coset space $\Gamma\backslash G/H$, a geometric construction of (Harish-Chandra's) discrete series in [AS], a construction of nonzero harmonic forms related to discrete series for semisimple symmetric spaces in [TW], and so on. It is natural to expect that an existence result for pseudo-Riemannian homogeneous spaces could open a theory on harmonic analysis on such nice double coset spaces.

Conversely, a nonexistence result of compact Clifford-Klein forms is also interesting. In fact, in a special case where G is a semisimple Lie group with $\mathbb{R}$-rank $G \geqslant 2$

Received 14 November 1991.
Author partially supported by the Inamori foundation and DMS-9100383.