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ON THE AVERAGE PERIOD OF AN ELLIPTIC CURVE

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Introduction. In [BrMc], Brumer-McGuinness studied statistical properties of elliptic curves defined over $\mathbb{Z}$. One feature of their work was the use of a large computer-assisted count to support certain conjectures not only on the number of elliptic curves with |discriminant| at most t, but also on the average period of an elliptic curve with |discriminant| at most t. The assertion of interest in this paper concerns the periods. The claim is that the average period of a curve with positive discriminant should be $\sqrt{3/2}$ times the average period of a curve with negative discriminant.

In this paper an elliptic curve will be a plane curve defined over $\mathbb{Z}$ and in normal form

$$\begin{aligned} w^2 &= 4z^3 - m_1 z - m_2 \\ &= 4(z - e_1)(z - e_2)(z - e_3), \qquad \mathrm{Re}(e_1) \leqslant \mathrm{Re}(e_2) \leqslant \mathrm{Re}(e_3). \end{aligned}$$

One sets $m = (m_1, m_2) \in \mathbb{Z}^2$. Such a curve will be denoted as E_m in the following. Because the curve is defined over $\mathbb{R}$, either one or three of the e_i are real. If the latter occurs, then the real points on the curve lie in two topological components, exactly one of which is unbounded. The unique unbounded component will be denoted $E_m^u(\mathbb{R})$.

Remark. The usual notation for the normal form of an elliptic curve is $w^2 = 4z^3 - g_2 z - g_3$. □

Define

$$P(m) = 27m_2^2 - m_1^3,$$

$$J(m) = \frac{-m_1^3}{P(m)},$$

$$\omega(m) = \int_{E_m^u(\mathbb{R})} \frac{dz}{\sqrt{4z^3 - m_1 z - m_2}}.$$

Remark. The usual notation for the defining equation of the discriminant (resp. J invariant) is $P = c_4^3 - 27c_6^2$ (resp. $J = c_4^3/(c_4^3 - 27c_6^2)$). The above expression for P

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