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ESTIMATING HECKE EIGENVALUES OF SIEGEL MODULAR FORMS

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0. Introduction. A central problem in the theory of automorphic forms is to estimate the Hecke eigenvalues of cusp forms. Historically, such eigenvalues first arose as multiplicative remainder terms in formulas for the number of representations of integers by certain quadratic forms [Ra]. In this case the eigenfunctions are holomorphic cusp forms for congruence subgroups of the modular group, and the best possible estimate for their eigenvalues was obtained by Deligne when he proved the Ramanujan-Petersson conjecture [De].

In this paper we shall consider the case of holomorphic Siegel cusp forms of degree $n \geqslant 2$. As when $n = 1$, the Hecke eigenvalues here are closely connected to representations by a positive quadratic form of scalar multiples of a fixed form in n variables. In contrast to the case $n = 1$, however, only relatively weak bounds are known for these eigenvalues. Such bounds have previously been obtained by generalizing two classical approaches: the Rankin-Selberg method and the method of Poincaré series and Kloosterman sums. In the case $n \geqslant 2$, we shall go well beyond these methods by estimating the matrix coefficients of certain representations. For precise statements of our results, see formulas (1.7), (1.8), and (1.9) below.

For the sake of exposition, we shall restrict our attention to modular forms for the full Siegel modular group (with trivial multiplier system when $n = 2$), although the method employed generalizes considerably. We remark that Shahidi [Sh] has obtained good estimates for the Hecke eigenvalues of generic cusp forms on quite general groups. However, when $n \geqslant 2$, holomorphic Siegel modular forms are not generic; so his estimates do not apply here.

In the final section further results are given concerning the absolute convergence of certain L-functions and the Hecke eigenvalues of singular modular forms (see [A]).

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1. Classical approach for cusp forms. Let $G = GSp_{2n}$ be the group of symplectic similitudes with respect to

$$J = \begin{pmatrix} 0 & -1_n \\ 1_n & 0 \end{pmatrix}$$

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