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SPECTRAL CONVERGENCE ON DEGENERATING SURFACES

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1. Introduction. The study of the spectrum of the Laplace operator has produced an extensive literature. (See [Cha] and the references therein.) Of special interest to recent applications has been the behavior of spectra on two-dimensional surfaces with degenerating metrics; for example, the case of hyperbolic metrics on Riemann surfaces is already quite complicated. (See [Hj], [Ji], [W1], [W2].) In this paper we show that, for a wide variety of degenerating metrics which have, however, quite different behavior from that of the hyperbolic metric, the spectrum converges to the spectrum of the surface with the degenerate metric.

Specifically, we consider surfaces M_0 with a singular metric, where the singularity in local coordinates is quasi-isometrically a cone. (See Sect. 2 for our model.) Such singularities were studied first by Cheeger [Che1] and subsequently by various authors, particularly in the context of $\bar{\partial}$, Dirac, and other first-order operators. (See [Chou], [BS], [S1], [S2].) It is a fundamental fact about metrics with cone singularities that the Laplacian Δ_0 on M_0 still has a discrete spectrum $\mathrm{Spec}(\Delta_0) = \{\lambda_i(0)\}_{i=0}^{\infty}$, which we order $0 = \lambda_0(0) < \lambda_1(0) \leqslant \lambda_2(0) \leqslant \ldots$. The natural question which then arises is the following: suppose we are given compact surfaces M_t with degenerating metrics g_t converging as $t \to 0$ to a metric on M_0 which has a cone singularity p. The singularity is assumed to be a double point; that is, locally we have two cones joined at their vertices. The noncompact surface $M_0 \backslash \{p\}$ may or may not be connected, and we shall refer to these two possibilities as the nonseparating and separating cases, respectively. We are interested in when Spec $(\Delta_t) = \{\lambda_i(t)\}_{i=0}^{\infty}$ converges to $\mathrm{Spec}(\Delta_0)$. To state the results precisely, we fix some notation: let $\{\varphi_i(t)\}_{i=0}^{\infty}$ denote a complete orthonormal basis of eigenfunctions with eigenvalues $\lambda_i(t)$, and for $\lambda > 0$ define the kernel function

$$K_t(x, y; \lambda) = \sum_{\lambda_i(t) < \lambda} \varphi_i(t)(x)\varphi_i(t)(y).$$

By spectral convergence we mean the following:

(∗) *Spectral convergence*

(i) For all $i \geqslant 1$, $\lim_{t\to 0} \lambda_i(t) = \lambda_i(0)$;

(ii) for any sequence $t_j \to 0$ there exists a subsequence $t_j' \to 0$ such that for all $i \geqslant 1$

$$\lim_{j\to\infty} \varphi_i(t_j') = \varphi_i(0)$$

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