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METRIC CURVATURE, CONVERGENCE, AND TOPOLOGICAL FINITENESS

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We use purely metric techniques to generalize the original convergence theorem of Gromov ([GLP]) and Cheeger's finiteness theorem ([C]) by replacing an upper curvature bound and lower volume bound with a lower bound on injectivity radius and by removing smoothness assumptions as well. By working with classes of metric spaces we are able to state our main result as a *compactness theorem* which contains the corresponding convergence or weak compactness statements for Riemannian manifolds.

This paper begins with conditions on inner metric spaces under which metric curvature bounds, geodesic completeness, and angles between minimal curves are preserved in Gromov-Hausdorff limits. In particular, we show that if $X_i \to X$ and the "comparison radius" for bounded curvature in each X_i is uniformly bounded below by some $r > 0$, then X has the same curvature bound, with comparison radius at least r. If in addition the X_i are geodesically complete, then X is also geodesically complete, provided the injectivity radii $\mathrm{inj}(X_i)$ have a uniform positive lower bound.

Aside from their use in the present paper, these general convergence results make possible the following technique of "metric pinching": Suppose X is a Riemannian manifold (or class of Riemannian manifolds) about which one would like to prove a diffeomorphism pinching theorem. The first step in solving the problem metrically is to give purely *metric* conditions $\mathscr{C}$ which characterize X up to isometry. The next step is to weaken those conditions to a new set of conditions, say $\mathscr{C}(\varepsilon)$. The conditions $\mathscr{C}(\varepsilon)$ should be such that, if $\{X_i\}$ is a Gromov-Hausdorff convergent sequence of spaces and X_i satisfies $\mathscr{C}(\varepsilon_i)$ with $\varepsilon_i \to 0$, then Propositions 5–9 below can be used to show that $\lim X_i$ satisfies $\mathscr{C}$. If the spaces X_i are Riemannian manifolds having fixed lower bounds on volume and curvature, then Yamaguchi's pinching theorem ([Y]) implies that for large enough i there is a diffeomorphism (and almost isometry), from X_i to X. As a final step in the solution, it may be desirable to find analytical conditions which imply $\mathscr{C}(\varepsilon)$ in order to improve the applicability of the theorem. This strategy was carried out in [P3] and [P4].

Another consequence of Propositions 5–7, together with the precompactness theorem proved in [P2] is that, if $\mathscr{M}(n, k, D, \varepsilon)$ denotes the class of n-dimensional, metrically and geodesically complete inner metric spaces of curvature $\geqslant k$, diameter $\leqslant D$, and injectivity radius $\geqslant \varepsilon > 0$, then $\mathscr{M}(n, k, D, \varepsilon)$ is Gromov-Hausdorff compact. The latter part of the present paper is devoted to obtaining topological

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