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TIME-DEPENDENT APPROACH TO RADIATION CONDITIONS

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1. Introduction. We study the momentum of scattering orbits for large time in classical as well as in quantum mechanics. The potential $V(x)$, $x \in \mathbb{R}^n$, is assumed to be smooth and to have the property

$$|\partial_x^\alpha V(x)| \leqslant C_\alpha \langle x \rangle^{-\varepsilon_0 - |\alpha|}$$

for any multi-index α. Here $0 < \varepsilon_0 < 1$ and $\langle x \rangle = (1 + |x|^2)^{1/2}$.

As a motivation, we recall that momentum p and position x become parallel. Explicitly, as can easily be proved, on any classical scattering orbit with energy λ,

$$p = \sqrt{\lambda}\frac{x}{|x|} + O(t^{-\varepsilon_0}) \qquad \text{for } t \to +\infty .$$

A "better approximation" is given as follows. Let $S(x, \lambda)$ be a solution to the eikonal equation $|\nabla S(x, \lambda)|^2 + V(x) = \lambda$, which behaves like $\sqrt{\lambda}|x|$ at infinity. Let $\gamma(\lambda) = p - \nabla S(x, \lambda)$. Then on any orbit

$$\gamma(\lambda) = O(t^{-1}) \qquad \text{for } t \to +\infty . \tag{1.1}$$

Moreover, we shall prove that

$$\nabla S(x, \lambda)\gamma(\lambda) = O(t^{-2}) \qquad \text{for } t \to +\infty . \tag{1.2}$$

These results are obtained by means of a simple differential inequality.

A natural quantization of $\nabla S(x, \lambda)$ is the pseudodifferential operator (Ps. D. Op.) $\overline{\nabla S}$ essentially obtained by symmetrizing the Ps. D. Op. with symbol $(\nabla S)(x, \xi^2 + V(x))$. Let $\overline{\gamma}^{\alpha(m)}$ denote an arbitrary product of m components of the Ps. D. Op. $\overline{\gamma} = p - \overline{\nabla S}$. With this definition we generalize (1.1) and (1.2) to the quantum mechanical case, in fact, with a very similar proof. We establish the analogous estimate

$$\langle x \rangle^{\ell} \overline{\gamma}^{\alpha(m)} e^{-itH} f(H) \langle x \rangle^{-m} = O(t^{-m+\ell+\varepsilon}) \tag{1.3}$$

for $t \to +\infty$ for any $\varepsilon > 0$ and ℓ with $m > \ell \geqslant 0$. Here $f \in C_0^\infty(\mathbb{R}^+)$, and $H = -\Delta + V$.

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