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TORSION ZERO-CYCLES AND ÈTALE HOMOLOGY OF SINGULAR SCHEMES

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0. Introduction. In this paper we prove a theorem which relates the torsion in the group of cycles of dimension zero modulo rational equivalence on a scheme which may be singular with its ètale homology group. It is a natural generalization of a theorem of Roitman. First, let us fix notations. Let k be an algebraically closed field and let $\mathscr{V}$ be the category of separate schemes of finite type over k. For $X \in Ob(\mathscr{V})$ let $CH_0(X)$ be the Chow group of cycles of dimension zero modulo rational equivalence on X as defined in [F]. We denote by $A_0(X)$ the subgroup of cycle classes of degree zero. We are interested in the torsion subgroup $A_0(X)_{tor}$ of $A_0(X)$. First, we recall the following theorem due to Roitman [R].

THEOREM. *Assume that X is proper and smooth over k. Then the Albanese map induces an isomorphism*

$$A_0(X)_{tor} \simeq Alb(X)_{tor}.$$

Here Alb(X) is the Albanese variety of X.

In this paper [B-1] Bloch provided a new proof of Roitman's theorem, at least restricted to the prime-to-p part, where $p = \mathrm{ch}(k)$. (The p-part is treated by Milne [M].) The essential ingredients are K-theoretic and cohomological methods (see [B-2, §4]) and the Weil conjecture proved by Deligne [D]. In our paper we observe that, combined with the ètale homology theory on a scheme (see [B-O]), Bloch's method is in fact powerful enough to deal with $A_0(X)_{tor}$ in a more general setting, where X is not necessarily supposed to be smooth over k. To state our theorem we fix a prime number $\ell \neq p$ and introduce the following condition for $Z \in Ob(\mathscr{V})$.

$$(*)_\ell \qquad \bigcap_{U \to Z} \mathrm{Ker}(H^1(Z, \mathbb{Q}_\ell) \to H^1(U, \mathbb{Q}_\ell)) = 0$$

where U ranges over all *regular* subschemes of Z and

$$H^1(*, \mathbb{Q}_\ell) = \left(\varprojlim_{\nu} H^1(*_{et}, \mathbb{Z}/\ell^\nu\mathbb{Z})\right) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell.$$

A sufficient condition for $(*)_\ell$ to hold will be given in Section 3. In particular, if

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