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PARAMETRIX CONSTRUCTION FOR A CLASS OF SUBELLIPTIC DIFFERENTIAL OPERATORS

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1. Introduction. The purpose of this paper is the construction of parametrices for certain nonnegative second-order operators. These operators are well approximated at a point by a Schrödinger operator with nonnegative polynomial potential. The essence of the symbol construction is obtaining uniform estimates for the fundamental solution of such equations and proving their smooth dependence as the polynomial varies.

We work with operators on $\mathbb{R}^n$ of the form

$$L(x, \partial_x) = A(x, \partial_{x'}) - \beta(x)\partial_{x_n}^2$$

where $\beta(x)$ is nonnegative and of finite type in $x' = (x_1, \ldots, x_{n-1})$; that is, $\sum_{|\alpha| \leqslant k} |\partial_{x'}^\alpha \beta(x)| \geqslant c_0 > 0$ for some c_0, k. Now $A(x, \partial_{x'})$ is assumed to be an elliptic differential operator in the variables $x' = (x_1, \ldots, x_{n-1})$, with nonnegative principle symbol, and its coefficients may depend on x_n.

If $\beta(x) = \alpha(x)^2$, where $\alpha(x)$ is of finite type ℓ (hence $k = 2\ell$), then the operator is of sum of squares type, and by results of Rothschild and Stein [RS], $L(x, \partial_x)$ is subelliptic with a gain of $2/(\ell + 1)$ derivatives on L^p Sobolev spaces. For positive β, subellipticity of $L(x, \partial_x)$ on L^2 follows from more general results of Oleinik and Radkevitch [OR], but with the derivative gain estimate less sharp than one expects from the sum of square case. Recently, Fefferman, Phong, and Sanchez-Calle have obtained estimates on the fundamental solutions for quite general positive second-order subelliptic operators, which imply the sharp regularity gain on L^p. (See [FP] and [FSc].)

The novelty of this paper is the realization of the parametrix as a pseudodifferential operator, where the principal symbol of the parametrix has an explicit homogeneous dependence on $\beta(x)$ and its derivatives to order k. The symbol is bounded in size by $\rho(x, \xi)^{-2}$ where

$$\rho(x, \xi) \approx |\xi'| + \sum_{|\alpha| \leqslant k} |\partial_{x'}^\alpha \beta(x) \xi_n^2|^{1/(|\alpha|+2)}.$$

As a result, one can essentially read off the following regularity theorem.

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