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ISOPERIMETRIC BOUND FOR λ_3/λ_2 FOR THE MEMBRANE PROBLEM

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1. Introduction. We consider the eigenvalues of the Dirichlet Laplacian on a bounded domain $\Omega \subset \mathbb{R}^2$ as defined by the equations

$$-\Delta u = \lambda u \quad \text{on } \Omega \quad \text{and} \tag{1.1a}$$

$$u = 0 \quad \text{on } \partial\Omega. \tag{1.1b}$$

We denote the eigenvalues by $\{\lambda_i\}_{i=1}^{\infty}$, listed in ascending order with multiplicities included so that

$$0 < \lambda_1 < \lambda_2 \leqslant \lambda_3 \leqslant \cdots. \tag{1.2}$$

We associate with these a corresponding sequence of orthonormal eigenfunctions $\{u_i\}_{i=1}^{\infty}$. In this paper we solve the problem of finding the optimal upper bound on the ratio λ_3/λ_2.

Our result in this paper is a second step toward solving an interesting conjecture of Payne, Pólya, and Weinberger that was made in their seminal paper [7] on eigenvalue ratios some thirty-five years ago. In this paper they conjectured that for the two-dimensional membrane problem, i.e., (1.1), the ratio λ_{m+1}/λ_m of consecutive eigenvalues is maximized over domains Ω and the index m, when $m = 1$ and Ω is a disk. They were able to show that, in any event, $\lambda_{m+1}/\lambda_m \leqslant 3$. In our recent papers [3, 4] we proved that λ_2/λ_1 is maximized by its value for a disk, $j_{1,1}^2/j_{0,1}^2 \approx 2.5387$. There seems to have been no work on λ_{m+1}/λ_m for $m > 1$ or even just on λ_3/λ_2 for the membrane problem between that of Payne, Pólya, and Weinberger and our own in this paper. We refer the reader to our introductions in [3] and [4] for a summary of related work during the intervening years. See also our references to these articles (particularly [4]) which include a nearly complete list of articles stemming from the original papers [6, 7] of Payne, Pólya, and Weinberger. (See also [1] and [10], of which we were unaware when our earlier articles were written.) We shall not repeat these references here, referring the reader instead to [4], but mention again that certain results of Chiti and Talenti concerning spherical rearrangements were instrumental to our proof. The Payne-Pólya-Weinberger result $\lambda_{m+1}/\lambda_m \leqslant 3$ was generalized to n dimensions (Dirichlet Laplacian on a bounded

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