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THE THIRD HOMOLOGY GROUP OF THE MODULI SPACE OF CURVES

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Let $M_{g,r}$ denote the moduli space parameterizing genus g Riemann surfaces together with r pairs (p_i, v_i) where p_i is a point of X and v_i is a nonzero tangent vector to X at p_i. It is known ([Harer 2]) that the homology group $H_k(M_{g,r})$ is independent of g and r for $g \gg k$ and that the stable rational cohomology contains a polynomial algebra on generators $\kappa_i \in H^{2i}$, $i = 1, 2, \ldots$ ([Miller], [Morita]). It is believed that this is all of $H_*M_{g,r}$ stably [Mumford 2], although the evidence for this claim is a bit weak. It is correct for $k = 1$ [Mumford 1] and $k = 2$ [Harer 1]; in this paper we will strengthen the evidence by proving that it is correct for $k = 3$ and giving a new proof for $k = 2$. Specifically, we will show (by techniques independent from [Harer 1]) the following.

THEOREM 0.

(a)
$$H_2(M_{g,r}; \mathbb{Q}) \cong \begin{cases} O & g = 2 \\ \mathbb{Q} & g \geqslant 3, \end{cases}$$

and for $r \geqslant 1$ there are natural maps

$$\beta_{ij}\colon M_{g,r} \to M_{g,r+1}$$

$$\alpha_i\colon M_{g,r} \to M_{g+1,r-1}$$

inducing isomorphisms on H_2 for $g \geqslant 3$.

(b)
$$H_3(M_{g,r}; \mathbb{Q}) = 0, \qquad g \geqslant 6.$$

The method we use to compute these groups is the following. Moduli space $M_{g,r}$ is the quotient of the Teichmüller space $T_{g,r}$ by the properly discontinuous action of the mapping class group $\Gamma_{g,r}$. As $T_{g,r}$ is topologically Euclidean space, $H_*(\Gamma_{g,r}; \mathbb{Q})$ is isomorphic to $H_*(M_{g,r}; \mathbb{Q})$. There is a triangulation of $T_{g,r}$ ([Harer 3]) obtained using certain special quadratic differentials due to Strebel ([Strebel]). By various combinatorial arguments ([Harer 2]) this may be used to construct two spherical,

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