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ON THE STRUCTURE OF THE CONFORMAL GAUSSIAN CURVATURE EQUATION ON $\mathbb{R}^2$

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1. Introduction. The purpose of this paper is to investigate the structure of the set of all solutions of the equation

$$\Delta u + Ke^{2u} = 0 \tag{1.1}$$

in $\mathbb{R}^2$, where $\Delta = \sum_{i=1}^{2} \partial^2/\partial x_i^2$ and K is a given nonpositive smooth function on $\mathbb{R}^2$. We shall establish the existence of a *unique maximal solution* of (1.1), and in some important cases we are able to *completely* classify *all* the solutions of (1.1). (Throughout this paper a solution of an equation shall always mean a solution in the classical sense.)

Equation (1.1) arises in Riemannian geometry. Let (M, g) be a Riemannian manifold of dimension 2 and K be a given function on M. The following question has been raised: can we find a new metric g_1 on M such that K is the Gaussian curvature of g_1 and g_1 is *conformal* to g (that is, $g_1 = \varphi g$ for some positive smooth function φ on M)? If we write $\varphi = e^{2u}$, then this is equivalent to the problem of solving the elliptic equation

$$\Delta_g u - k + Ke^{2u} = 0 \tag{1.2}$$

on M where Δ_g and k are the Laplace-Beltrami operator and the Gaussian curvature on M in the g-metric, respectively. In case M is compact, equation (1.2) has been considered by many authors, and we refer the reader to the recent monograph by Kazdan [K] for details and references. In case M is complete and noncompact, the first natural case seems to be $M = \mathbb{R}^2$ and g is the usual Euclidean metric. In this case, equation (1.2) reduces to equation (1.1).

The first result concerning equation (1.1) seems due to L. V. Ahl'fors [A] in 1938. He showed that in case $K \equiv -1$ (or any negative constant), equation (1.1) does not have any solution in the entire space $\mathbb{R}^2$. His result was later improved and extended by Wittich [W], Keller [Ke], Osserman [Os], Sattinger [S1], Oleinik [O], Ni [N1], McOwen [M], and Cheng and Lin [CL]. These results basically contain the following

THEOREM A. *If $K \leqslant 0$ in $\mathbb{R}^2$ and $K \leqslant -C|x|^{-2}$ near ∞, then equation* (1.1) *does not possess any solution on $\mathbb{R}^2$.*

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