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THE p-ADIC SIGMA FUNCTION

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§1. Introduction. The object of this article is to construct the *p-adic sigma function* for elliptic curves of *ordinary reduction* defined over fields complete for a discrete valuation with residue field of characteristic $p > 0$. We had promised (in fact, we have made use of) such a construction in our earlier papers: [M-T-T] and [M-T 2]. Here we fulfill that promise.

There has, of course, been ample precedent for the construction of p-adic sigma functions or more general theta functions in different contexts and for numerous purposes. One may cite Barsotti [Ba], Breen [Br], Cristante [Cr], McCabe [McC], Morikawa [Mo], Néron [Né 1, 2, 3] and Norman [No] for treatments of the theory of theta series in various p-adic settings.

Beginning, perhaps, with an unpublished handwritten manuscript of Abramov and Rosenblum, where a mod p version of the sigma function was used to define a mod p height, a number of constructions of p-adic sigma functions have been sought for the principal purpose of providing theories of canonical heights for points on elliptic curves; notably there is the work of Bernardi [Be], Perrin-Riou [P-R1, 2], and Bernardi, Goldstein and Stephens [B-G-S]. The connection with the canonical height is also our main motivation. We refer the reader to Chapter II of [M-T-T], where explicit formulas are given which connect sigma with the canonical p-adic height, and where the practical problem of computing sigma is discussed in some detail.

Let R be a complete discrete valuation ring with uniformizer π and residue field $k = R/\pi R$ of characteristic $p > 0$. Let K be the field of fractions of R, and let $\bar{R}$ be the integral closure of R in an algebraic closure $\bar{K}$ of K. Let $E_{/K}$ be an elliptic curve over K. Let E be the connected component of its Néron model over R and E^f the *formal group* of E. We suppose that E is *ordinary*, i.e., that, over $\bar{k}$, E^f is isomorphic to the formal multiplicative group $\mathbb{G}_m^f$. In that case there is a good p-adic analog of the classical Weierstrass sigma function if p is odd, and of its square if $p = 2$, which we discuss in this paper.

The classical sigma function is defined on the universal covering of the elliptic curve in question, i.e., is a many-valued function on the curve. In contrast, our p-adic σ is single-valued, but is in general defined only on the formal group E^f. As in the classical case, E determines σ only up to a constant, which is fixed once we choose a nonzero invariant differential ω on E. Let ω be such a differential which is finite and nonzero on the special fiber. We show (Theorem 3.1) that our $\sigma = \sigma_{E,\omega}$ enjoys, and is characterized by, any one of several properties which are analogs of those in

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