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A NOTE ON QUASI-DIAGONAL C^*-ALGEBRAS AND HOMOTOPY

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A complete set of obstructions to the quasi-diagonality of a C^*-algebra is still unknown. Various examples suggest that the obstructions are from noncommutative topology. In this note we prove that a C^*-algebra which is homotopically dominated by a quasi-diagonal C^*-algebra is quasi-diagonal (Theorem 5). This, clearly, confirms the topological nature of quasi-diagonality.

Let us also mention that our result improves over a result of Eberhard Kirchberg ([4]), who had shown that the suspension of a C^*-algebra has an extension by an ideal of block-diagonal compact operators, which is quasi-diagonal.

We begin by recalling some basic facts on quasi-diagonality. Quasi-diagonal operators on separable Hilbert spaces were defined by P. R. Halmos [3] as compact perturbations of block-diagonal operators. A generalization to sets of operators goes as follows. A set $\Omega \subset \mathscr{L}(H)$ (where $\mathscr{L}(H)$ denotes the bounded operators on the Hilbert space H) is quasi-diagonal if for every $\varepsilon > 0$ and finite subsets $\omega \subset \Omega$ and $\chi \subset H$ there is a finite-rank orthogonal projection P such that $\|[P, T]\| < \varepsilon$ if $T \in \omega$ and $\|(I - P)h\| < \varepsilon$ if $h \in \chi$. This is equivalent to the apparently stronger requirement where $\|(I - P)h\| < \varepsilon$ is replaced by $Ph = h$ for $h \in \chi$.

A C^*-algebra $\mathscr{A}$ is quasi-diagonal if there is a faithful representation ρ such that $\rho(A)$ is a quasi-diagonal set of operators. In this definition $\mathscr{A}$ is not necessarily unital, but it is easily seen that $\mathscr{A}$ is quasi-diagonal if and only if $\tilde{\mathscr{A}}$ is quasi-diagonal, where $\tilde{\mathscr{A}}$ is the C^*-algebra obtained by adjoining a unit to $\mathscr{A}$.

It is a consequence of our noncommutative Weyl-von Neumann type theorem [7] that a separable C^*-algebra is quasi-diagonal if and only if for every faithful representation π of A, such that $\pi(A)$ does not contain nonzero finite-rank operators, $\pi(A)$ is a quasi-diagonal set of operators. This fact can be used to show that a C^*-algebra is quasi-diagonal if and only if all its finitely generated sub-C^*-algebras are quasi-diagonal. This in turn shows that the separability condition can be dropped in the preceding assertion.

It will be useful to derive some equivalent quasi-diagonality conditions.

1. THEOREM. *Let A be a unital C^*-algebra. The following are equivalent.*

(i) *A is quasi-diagonal.*
(ii) *Given $\varepsilon > 0$ and $F \subset A$ a finite subset, there is a unital completely positive map $\varphi: A \to B$, where B is a finite-dimensional C^*-algebra, such that*

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