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ON EXPLICIT INTEGRAL FORMULAS FOR $GL(n, \mathbb{R})$-WHITTAKER FUNCTIONS

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1. Introduction. The introduction by Selberg [10] of the techniques of harmonic analysis to the study of number theoretic problems brought about a new interest in the theory of special functions. Since Selberg's seminal work on this subject, various types of special functions relevant to number theory have been studied by numerous authors.

Of particular interest are the functions that arise as the Fourier coefficients of automorphic forms on a reductive group—that is, the "Whittaker functions" (so named because, for the group $GL(2, \mathbb{R})$, the special functions that occur correspond to functions originally considered by E. T. Whittaker). The systematic study of Whittaker functions was initiated by Jacquet [7], who produced for them certain integral representations and then proved their analytic continuation and functional equations.

In this paper, we wish to consider the Whittaker functions $W_{(n,\nu)}(y)$ (cf. §1) associated with nonramified principal series representations of the group $GL(n, \mathbb{R})$. Such Whittaker functions have many number-theoretic applications, in particular to the problem of "local calculations at infinity" arising from the study of automorphic L-functions via the Rankin-Selberg method.

Specifically, this method leads (at the archimedean places) to certain integrals or "Mellin transforms" of these Whittaker functions. One expects such integrals to be expressible as products of Gamma functions; expressions of this type should in turn yield the functional equations at infinity for the L-functions in question.

Explicit knowledge of $GL(n, \mathbb{R})$-Whittaker functions has so far been insufficient to carry out the above local calculations in any generality. In this article, we will develop new integral representations for $W_{(n,\nu)}(y)$ that will be applicable to just such calculations.

Theorem 2.1 below contains our principal result, which is the rather striking fact that the $GL(n, \mathbb{R})$-Whittaker function $W_{(n,\nu)}(y)$ may be written as an $(n-2)$-fold integral involving the corresponding $GL(n, \mathbb{R})$-Whittaker function (and some Bessel functions). Another new formula that may be relevant to applications is given in our Proposition 5.1, where $W_{(n,\nu)}(y)$ is expressed as an integral involving only Bessel functions (of somewhat more complicated arguments).

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