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ON ARTIN'S CONJECTURE AND THE CLASS NUMBER OF CERTAIN *CM* FIELDS, II

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0. Introduction. Let K be a CM field and let k be its maximal totally real subfield with $n = [k:\mathbb{Q}]$. For any number field M we let D_M denote the absolute value of the discriminant of M. Then $D_K = D_k^2 f$, where f is the norm of the relative different. The problem of finding an effective lower bound for h_K, the class number of K, is directly related to bounding the distance of a real zero of the zeta function $\zeta_K(s)$ from 1. The first breakthrough in this direction was the Brauer-Siegel theorem, but the class number estimates that follow from it are weak and ineffective.

A major improvement was introduced by Stark [8] and then strengthened by Odlyzko [6] and Hoffstein [1]. In these papers, strong and effective class number bounds were obtained for $n > 2$, with the restricton that k be normal over $\mathbb{Q}$ or attainable by a sequence of normal extensions. However, in the case of general k these results leave a crucial $n!$ in a denominator which has the effect of considerably weakening the final (still effective) class number bound when discriminants are small. In particular, because of this $n!$ the above-mentioned results imply that $h_K \to \infty$ as $n \to \infty$ only if one makes the additional restriction that $D_k > (Cn)^{2n}$ for some sufficiently large C.

As noted above, the case where k is normal is the "best possible case." In this paper and in [2], we work at the other extreme in what is essentially the worst possible case, i.e., the case where k is as "far away" from being normal over $\mathbb{Q}$ as possible. In particular, let $\mathcal{S}$ be the set of all totally real fields k with the property that the Galois group of the Galois closure of k over $\mathbb{Q}$ is S_n, where $n = [k:\mathbb{Q}]$. Then the following theorem is proven by the authors in [2]:

THEOREM. *Let $k \in \mathcal{S}$ and let K be any totally complex quadratic extension of k that does not contain an imaginary quadratic field. If β is a real zero of ζ_K/ζ_k, then $1 - \beta > 1/(3n4^n \log(D_k f^{1/n}))$.*

The previous theorem essentially replaces the $n!$ in Stark's bound by 4^n, and as a result we can replace the $(Cn)^{2n}$ condition by one involving only C^n. In particular, one has

COROLLARY. *Let K, k be as above. For any $\delta > 0$ there exists an effective constant $C > 0$ such that when $D_k > C^n$, $h_K > (1+\delta)^n$.*

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