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THE RANK CONDITION AND CONVERGENCE OF FORMAL FUNCTIONS

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Introduction. In this paper we treat two facts concerning convergence of formal holomorphic functions. One is that, for a germ of holomorphic mapping with maximal generic rank, the formal function at the target converges if its pullback does so (Gabrielov). The other is propagation of convergence along an exceptional set.

Let $\Phi: Y \to X$ be a morphism between complex spaces and $\varphi: A = \mathcal{O}_{X,\xi} \to B = \mathcal{O}_{Y,\eta}$ the induced homomorphism between analytic local algebras. To simplify the problem, assume that the germs X_ξ and Y_η are reduced and irreducible, i.e., A and B are integral domains. The author has been interested in equivalence of the following conditions:

(0) $\exists a \geqq 1, \exists b \geqq 0, \forall f \in A: av(f) + b \geqq v(\varphi(f))$, *where* $v(f)$ *denotes the vanishing order of* f *at the point* (*see* §2 *for the definition of* $v(f)$).
(1) *Gabrielov's strong rank condition*: *The image* $\Phi(Y_\eta) \subset X_\xi$ *has full topological dimension, i.e.,* $\operatorname{grk} \varphi = \dim X_\xi$.
(2) $\hat{\varphi}^{-1}(B) = A$, *i.e.,* $f \in \hat{A}$ *is convergent if its pullback is convergent, where* $\hat{\varphi}: \hat{A} \to \hat{B}$ *denotes the canonical extension to the completions.*
(3) φ *is a closed embedding with respect to the Krull topology.*
(4) φ *is injective and* $\varphi(A)$ *is closed in* B.

These are fundamental properties in the study of general analytic map germs. Gabrielov [G2] very early asserted (1) ⇒ (2) but his proof was very difficult. (Indeed, he has proved that even the weaker rank condition $r_1 = r_2$ implies (1) and (2). See §10.) Later, in the case A is regular, Moussu-Tougeron [MT] and Malgrange [M2] proved (1) ⇒ (2) and Eakin-Harris [EH] proved (1) ⇔ (2) independently. In general case, applying both [G2] and [EH], Milman [Mi] proved (1) ⇐ (3) and Becker-Zame [BZ] proved (4) ⇒ (1) (see (9.4)). (2) ⇒ (3) is well known and (3) ⇒ (4) is trivial. (0) ⇔ (1) is proved by the present author [I2], [I3]. The main purpose of this paper is to give a new and simpler geometric proof of the following:

THEOREM 1 (see (9.2)). (1) ⇔ (2) (⇔ (3)).

Hence we can see the equivalence of the conditions (0)–(4) not using the difficult part of [G2].

Now let E be an exceptional set in a reduced complex space X and $\hat{\mathcal{O}}$ the completion of the structure sheaf $\mathcal{O}$ along E. We assume that X is irreducible along

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