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THE INFINITESIMAL ABEL-JACOBI MAPPING AND MOVING THE $\mathcal{O}(2) + \mathcal{O}(-4)$ CURVE

HERBERT CLEMENS

0. Introduction. If C is a smooth rational curve embedded in a smooth projective surface S, then there are only two possibilities:

(1) $C^2 < 0$, in which case the conormal bundle $\mathcal{I}_C/\mathcal{I}_C{}^2$ is positive, so that $H^1(\mathcal{I}_C{}^n/\mathcal{I}_C{}^{n+1}) = 0$ and global sections of $\mathcal{I}_C$ on a formal neighborhood of C give a contraction $S \to S_0$ which maps C to a point and takes (S-C) isomorphically onto the rest of S_0. ([2] is a general reference for formal contraction criteria.)

(2) $C^2 \geqslant 0$, in which case $h^0(N_{C/S}) > 0$ and $h^1(N_{C/S}) = 0$. So C must move in S.

Suppose now that a smooth rational curve C is embedded in a smooth threefold V. One can ask the analogous question, is it always true either that C contracts or that C, or perhaps some multiple of C, moves? The purpose of this paper is to answer that question in the negative.

Our method will be to choose a situation in which the normal bundle $N_{C/V}$ is incompatible with the existence of a contraction, namely, a situation in which K_V is trivial but $H^1(N_{C/V}{}^*) \neq 0$. (See [4, Lecture 16].) We find a family of such V and show that the monodromy representation on $H^3(V)$ is irreducible for this family, so that the Abel-Jacobi mapping for deformations of mC in V must be trivial for generic V. On the other hand, we specialize V to a (singular) V_0 for which the derivative of the (generalized) Abel-Jacobi map is nonzero, thereby obtaining a contradiction of the assumption that deformations of mC exist for generic V.

Now $N_{C/V}{}^* = \mathcal{O}(a) + \mathcal{O}(b)$, $a + b = 2$, so there are only three cases in which $H^1(N_{C/V}{}^*) = 0$. Work of H. Laufer and M. Reid shows that C either contracts or moves in two of these cases. Only the case $a = -1$, $b = 3$, remains to be settled.

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1. The infinitesimal Abel-Jacobi mapping. We begin by discussing a relation between the Hodge theory of a complex projective variety V and families of subvarieties of V.

An algebraic family F of subvarieties of V is given by a subvariety $E \subseteq F \times V$ such that restriction of the first projection, $p: E \to F$, is a proper morphism to an

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