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CONDUCTOR, DISCRIMINANT, AND THE NOETHER FORMULA OF ARITHMETIC SURFACES

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0. Introduction. In the classical theory of ramification, there is a fundamental equality *conductor = discriminant*. The purpose of this paper is to establish this equality for relative curves.

The *conductor* is an integer defined for arbitrary scheme of finite type over a discrete valuation ring with perfect residue field by using ℓ-adic étale cohomology. On the other hand, the *discriminant* was defined only for finite extension of discrete valuation rings. Recently, Deligne defined a canonical isomorphism between (power of) the determinant invertible sheaves of higher direct images of the sheaves of (bi-) differentials of proper smooth curves and called it the discriminant [D]. We define an integer discriminant using this isomorphism. Roughly speaking, we may say that the discriminant is the intersection number with the infinite divisor of the moduli space of stable curves. And then we prove the equality *conductor = discriminant*.

Let S be the spectrum of a henselian discrete valuation ring with algebraically closed residue field and consider a proper flat regular S-scheme $f\colon X \to S$ with smooth generic fiber. The Artin conductor of X over S is the integer defined by

$$\mathrm{Art}(X/S) = \chi(X_{\bar{\eta}}) - \chi(X_s) + Sw_S H^*(X_{\bar{\eta}}, \mathbb{Q}_\ell),$$

where $Sw_S H^*(X_{\bar{\eta}}, \mathbb{Q}_\ell)$ is the alternating sum of the Swan conductor.

We consider the discriminant. We review the classical case where X is finite over S. In this case, there is a canonical homomorphism, which is an isomorphism on the generic point, of invertible $\mathcal{O}_S$-modules

$$\Delta\colon (\det f_*\mathcal{O}_X)^{\otimes 2} \to \mathcal{O}_S;$$

$$(x_1 \wedge \cdots \wedge x_n) \otimes (y_1 \wedge \cdots \wedge y_n) \mapsto \det\big(Tr_{X/S}(x_i y_j)\big),$$

where n is the degree of X over S. Then the discriminant is defined to be the order of Δ. Here the order means the length of the cokernel. In this case, it is well known that the equality *conductor = discriminant*, i.e., $\mathrm{Art}(X/S) = \mathrm{ord}\,\Delta$, holds.

Now consider the case where X is a geometrically connected curve over S, which is our main interest. We define the discriminant in this case in a similar way as above. For this purpose, we need the following result of Deligne ([D],

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