Vol. 57, No. 1 DUKE MATHEMATICAL JOURNAL © August 1988

AN ANALOGUE FOR HARMONIC FUNCTIONS OF KOLMOGOROV'S LAW OF THE ITERATED LOGARITHM

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Ever since Kolmogorov proved his celebrated **law of the iterated logarithm** (LIL) for independent random variables, efforts have been made to prove this result in the analysis setting. Some of the existing results are described in [3]. They include the LIL for lacunary series of R. Salem and A. Zygmund [26], P. Erdös and I. Gál [11], M. Weiss [31], and S. Takahashi [30], as well as the recent LIL of N. G. Makarov [20] for functions in the Bloch class and the more general result for arbitrary analytic functions proved in R. Bañuelos [2], which follows from a subgaussian estimate due to A. Chang, M. Wilson, and T. Wolff [7]. We should also mention here the recent upper-half LIL for functions in the Zygmund class of J. Anderson and L. Pitt [1]. The purpose of this paper is to prove an upper-bound case of an LIL for arbitrary harmonic functions in the upper half space. With regard to the upper estimate, this result is more general than any of the LILs mentioned above. Before we discuss this more precisely, let us recall the classical theorem of Kolmogorov which, in the words of K. L. Chung [9], page 231, "is a crowning achievement in classical probability theory."

THEOREM (Kolmogorov [19]). *Let $\{X_n\}_{n=1}^{\infty}$ be a sequence of independent random variables with mean zero and variance one. Suppose $|X_n| \leqslant \varepsilon_n n/\sqrt{\log\log n}$ for some constants $\varepsilon_n \to 0$. Then for almost every ω*

$$\limsup_{n\to\infty} \frac{S_n(\omega)}{\sqrt{2n\log\log n}} = 1, \tag{0.1}$$

where

$$S_n = \sum_{k=1}^{n} X_k.$$

Kolmogorov's LIL has been generalized in many directions with applications in such fields as statistics, number theory, differential equations, the study of Brownian motion on manifolds, dynamical systems, ergodic theory, and very

Received July 30, 1987. Revision received September 11, 1987. Rodrigo Bañuelos supported by an NSF postdoctoral fellowship. Ivo Klemes supported by a joint NSERC and I. W. Killam postdoctoral fellowship. Charles N. Moore supported in part by NSF.