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THE ERROR TERM IN NEVANLINNA THEORY

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In the sixties I conjectured that Roth's theorem could be improved to an inequality of type

$$\left|\alpha - \frac{p}{q}\right| \geqq \frac{1}{q^2(\log q)^{1+\varepsilon}}.$$

In other words, given ε and α algebraic, the inequality above holds for all but a finite number of fractions p/q in lowest form. (Cf. [La 6], [La 1], [La 4].) The inequality can also be written in the form

$$-\log\left|\alpha - \frac{p}{q}\right| - 2\log q \leqq (1 + \varepsilon)\log\log q.$$

Osgood ([Os 1], [Os 2]) has pointed out that the 2 in Roth's theorem corresponds to the 2 in Nevanlinna's inequality for meromorphic functions. But Vojta had the extraordinary insight that the second main theorem (SMT) of Nevanlinna theory, in both its one-dimensional and higher-dimensional versions, has an analogous statement in the theory of heights, which gives rise to a major conjecture in diophantine analysis ([Vo 1], [Vo 2], [Vo 3]). It occurred to me to transpose my conjecture about the "error term" in Roth's theorem to the context of this higher-dimensional analytic theory and see whether it held there.

I shall deal here with the Stoll–Carlson–Griffiths theorem in higher dimensions ([St], [C–G], [Gr]), which contains as a special case Nevanlinna's own theorem for $\mathbf{P}^1$, i.e., for meromorphic functions on $\mathbf{C}$. In the dictionary, the Nevanlinna function T_f corresponds to the number-theoretic height $\log q$. I shall prove the analogue of the exponent $3/2 + \varepsilon$ instead of the $1 + \varepsilon$ in this analytic context. (I am unable to get the $1 + \varepsilon$ exactly.) Even in the most classical case of meromorphic functions, I believe this estimate is new and improves both the error term in SMT and Nevanlinna's estimate in the theorem on the logarithmic derivative, where Nevanlinna's constants are replaced by $3/2 + \varepsilon$. However, the problem remains as to whether one has $1 + \varepsilon$ in general, and what is the best possible result in general. Thus the error term has a structure of its own, which should also occur in Vojta's conjecture.

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