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REPRESENTATIONS OF sl(n, $\mathbb{C}$) AND THE TODA LATTICE

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Introduction. The primary purpose of this paper is to show that the representation-dependent procedure of van Moerbeke and Mumford [20] for transforming the flow of the Toda lattice equations to a linear flow on the Jacobian of the representation-dependent spectral curve is independent of the Lie algebra representation used in that these linear flows are all locally injective homomorphic images of one another on isogenous Abelian varieties. In addition, we give a criterion that the linearization procedure of van Moerbeke and Mumford work for an arbitrary matrix Lax equation involving eigenspaces of arbitrary dimension. Lastly, as an interesting by-product of a detailed example, we find an example of two algebraic curves with isogenous but nonisomorphic Jacobians. In this introduction we explain the problem and then describe our solution and the additional contents of this paper.

The periodic Toda lattice is an example of a completely integrable system; namely, it is the Hamiltonian system in $(q_1, \ldots, q_n, p_1, \ldots, p_n)$

$$\frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i} \quad \text{and} \quad \frac{dq_i}{dt} = \frac{\partial H}{\partial p_i} \tag{0.1}$$

with Hamiltonian

$$H(q, p) = \frac{1}{2} \sum_{i=1}^{n} p_i^2 + \sum_{i=1}^{n} e^{q_i - q_{i+1}} \quad (\text{where } q_{n+1} \text{ means } q_1),$$

and it possesses n integrals in involution, enough to evolve as a linear flow on a torus.

The Toda lattice fits into the framework of a general Lie algebra integrability theorem of Adler [1], Kostant [12], and Symes [17] that says roughly that a vector space decomposition of a Lie algebra into two subalgebras $L = K \oplus N$ leads to a Hamiltonian system that has many integrals and takes the Lax form

$$\frac{dA}{dt} = [A(t), B(t)] \qquad A, B \in L. \tag{0.2}$$

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