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IRREGULARITY OF SURFACES WITH A LINEAR PENCIL

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In the study of projective complex surfaces, especially of those of general type, it is often interesting to determine the numerical invariants of a given surface S. Among these invariants, one has in particular the Euler characteristic $\chi(\mathcal{O}_S)$, the geometric genus $p_g = \dim H^0(\omega_S)$, and the irregularity $q = \dim H^1(\omega_S)$; they are related by the equation $\chi(\mathcal{O}_S) = 1 - q + p_g$. Due to this equation, there are only two independent invariants among the above three, and in some cases, the irregularity q can be determined (e.g., simply connected surfaces has $q = 0$), so that only one invariant between p_g and $\chi(\mathcal{O}_s)$ has to be known. In more general cases, although one is unable to know the exact value of q, it is possible to have an upper bound for it, which will serve to get a control of the difference between p_g and $\chi(\mathcal{O}_S)$. An important one of such situations is the following.

Let S be a (smooth complex projective) surface, with a fibration $f\colon S \to C$ over a smooth curve C of genus b. Let g be the genus of a general fibre F of f. Then it is well known that $b \leqq q \leqq b + g$, and when $g \geqq 2$, we have $q = b + g$ if and only if f is a trivial fibration. Therefore one can ask the following natural question: if f is not trivial, what will be the best upper bound of q?

Now on the one hand, we have easy examples with $q = \frac{1}{2}(g + 1) + b$ (Example 1 below); on the other hand, we have shown in [7], Corollary 3 to Theorem 2, the inequality

$$q \leqq \tfrac{1}{6}(5g + 1) + b$$

for nontrivial fibrations. But it is unlikely that this inequality gives the best bound for q, since its proof is not very accurate. In the present article, we shall consider the case where $b = 0$, with the following main theorem.

THEOREM 1. *Let S be a nonruled surface with a linear pencil Λ of curves of (geometric) genus g. Then*

$$q \leqq \tfrac{1}{2}(g + 1).$$

This theorem will be shown in §1. Then in §2, some examples are given to show that the given inequality is sharp. Finally, we use this theorem to get an

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