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A SHARP CASTELNUOVO BOUND FOR SMOOTH SURFACES

ROBERT LAZARSFELD

Introduction. Consider a smooth irreducible complex projective variety $X \subset \mathbb{P}^r$ of degree d and dimension n, not contained in any hyperplane. There has been a certain amount of interest recently in the problem of finding an explicit bound, in terms of n, d and r, on the degrees of hypersurfaces that cut out a complete linear system on X. At least for $r \geqslant 2n + 1$, the best possible linear inequality would be:

$$(*) \qquad H^1\big(\mathbb{P}^r, I_{X/\mathbb{P}^r}(k)\big) = 0 \quad \text{for } k \geqslant d + n - r.$$

This was established for (possibly singular) curves in [GLP], completing classical work of Castelnuovo [C]. For X of arbitrary dimension, Mumford (cf. [BM]) showed that $H^1(I_{X/\mathbb{P}^r}(k)) = 0$ for $k \geqslant (n + 1)(d - 2) + 1$. Pinkham [P] subsequently obtained the sharper estimate that if X is a surface, then hypersurfaces of degree $\geqslant d - 2$ [resp. $\geqslant d - 1$] cut out a complete series when $r \geqslant 5$ [resp. $r = 4$], but he left open the question of whether or not (*) holds. Recently Gruson has extended Pinkham's theorem to threefolds.

The purpose of this note is to complete Pinkham's result by establishing the optimal bound (*) for surfaces:

THEOREM. *Let $X \subset \mathbb{P}^r$ be a smooth irreducible complex projective surface of degree d, not contained in any hyperplane. Then hypersurfaces of degree $d + 2 - r$ or greater cut out a complete linear series on X.*

By the theory of Castelnuovo–Mumford [M, Lecture 14], the theorem has implications for the equations defining X in $\mathbb{P}^r$:

COROLLARY. *In the situation of the theorem, the ideal sheaf $I_{X/\mathbb{P}^r}$ is $(d + 3 - r)$—regular in the sense of Castelnuovo–Mumford. In particular, the homogeneous ideal of X is generated by forms of degrees $(d + 3 - r)$ or less.*

Bounds on the regularity of an ideal sheaf are important in connection with algorithms for computing syzygies (cf. [BS]), and this accounts for some of the recent interest in these questions.

The proof of the theorem revolves around a technique used by Gruson and Peskine in their work on space curves [GP1, GP2]. As in the arguments of

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