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DETERMINANTS OF LAPLACIANS AND A SECOND LIMIT FORMULA IN GL(3)

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0. Introduction. Let D be a positive, self-adjoint, elliptic operator on L^2 of a closed, connected Riemannian manifold, with eigenvalues $0 \leqslant \lambda_0 \leqslant \lambda_1 \leqslant \cdots$. To such a D associate the Dirichlet series

$$\zeta_D(s) = \sum_j \frac{1}{\lambda_j^s}$$

(where $\lambda = 0$ is excluded if it occurs). It was shown in [MP] that $\zeta_D(s)$ converges for s with $\mathrm{Re}(s)$ large, and admits a meromorphic continuation to the entire plane, which is regular at $s = 0$. Because of the formal identity

$$-\log \prod_j \lambda_j = \frac{d}{ds}\zeta(s)\Big|_{s=0}$$

the determinant of D is defined as the regularized value

$$\det(D) = e^{-\zeta_D'(0)}.$$

The study of such determinants has recently undergone development due to their occurrence in Polyakov's string theory [P]. Two cases of interest are those of the Laplace operator on Riemann surfaces, and the (square of the) Dirac operator on Riemann surfaces with spin structures. They correspond to bosonic and fermionic string theories, as explained in [AMV].

For surfaces of genus $g > 1$ these have recently been treated by D'Hoker and Phong [DP] and Sarnak [Sar], who were able to express the determinants as special values of Selberg zeta functions.

In his recent M.I.T. thesis [K], Kierlanczyk studied the $g = 1$ case of the two dimensional tori. The determinant of the Laplace operator can in this case be expressed in terms of Eisenstein series on the upper half plane, and that of the Dirac operator is a shifted version of it. Kierlanczyk exploits the first and second limit formula of Kronecker to compute these. His method is closely related to that of Ray and Singer [RS] in their computation of analytic torsion on 2-tori.

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