Vol. 55, No. 2 DUKE MATHEMATICAL JOURNAL © June 1987

A THEOREM ON REFINING DIVISION ORDERS BY THE REVERSE LEXICOGRAPHIC ORDER

DAVID BAYER AND MICHAEL STILLMAN

Let k be an infinite field of any characteristic, and let $S = k[x_1, \ldots, x_n]$ be a graded polynomial ring, where each x_i has degree one. Let $I \subset S$ be a homogeneous ideal.

Let S_d denote the finite vector space of all homogeneous, degree d polynomials in S, so $S = S_0 \oplus S_1 \oplus \cdots \oplus S_d \oplus \cdots$. Writing I in the same manner as $I = I_0 \oplus I_1 \oplus \cdots \oplus I_d \oplus \cdots$, we have $I_d \subset S_d$ for each d. An order $>$ on the monomials of S_d for each d is compatible with the monoid structure on the monomials of S if whenever $x^A > x^B$ for two monomials x^A, x^B, then $x^C x^A > x^C x^B$ for all monomials x^C. We shall only consider orders satisfying this compatibility condition.

If an order $>$ is a strict order on the monomials of each degree, one can use $>$ in applying the division algorithm to constructing a standard (Gröbner) basis for I. The standard basis for I, and its properties, will vary in a crucial way with the choice of order $>$. The subject of computing standard or Gröbner bases has a long history; see [Bay85] for a recent survey.

One can generalize the necessary definitions to nonstrict orders $>$, which fail to distinguish between all monomials of a given degree: For each polynomial $f \in S$, define $\mathrm{in}(f)$ to be the sum of those terms cx^A of f which are greatest with respect to the order $>$. Define $\mathrm{in}(I)$ to be the ideal generated by $\{\mathrm{in}(f) | f \in I\}$. Define $f_1, \ldots, f_r$ to be a standard basis for I with respect to the order $>$ if $\mathrm{in}(f_1), \ldots, \mathrm{in}(f_r)$ generate the ideal $\mathrm{in}(I)$. If $>$ is a strict order, $\mathrm{in}(I)$ will be a monomial ideal; if $>$ is not strict, $\mathrm{in}(I)$ may fail to be a monomial ideal.

A nonstrict order $>_1$ can be refined to a strict order by breaking any ties with a fixed strict order $>_2$; the resulting order $>_3$ is then a compatible order, so the usual division algorithm can be applied to compute standard bases with respect to $>_3$. Let $\mathrm{in}_1, \mathrm{in}_2, \mathrm{in}_3$ correspond to $>_1$, $>_2$, $>_3$. We shall see that $\mathrm{in}_3(I) = \mathrm{in}_2(\mathrm{in}_1(I))$, so a standard basis with respect to $>_3$ is already a standard basis with respect to $>_1$. Call $>_3$ the refinement of $>_1$ by $>_2$. Thus, refinements provide a mechanism for computing with nonstrict orders. This has been observed for example in [MoMö83], where in the affine setting, homogenizing bases (in the above sense, standard bases with respect to the total degree order) are computed via standard bases with respect to a strict order.

We recall two frequently used strict orders: The lexicographic order is defined by $x^A > x^B$ if the first nonzero entry in A-B is positive. The reverse lexico-

Received October 23, 1986.