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UNIFORMIZATION OF ATTRACTING BASINS FOR EXPONENTIAL MAPS

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Introduction. The complex exponential family $E_\lambda(z) = \lambda \exp(z)$, when considered in the context of iteration, provides a rich source of examples both in geometric function theory and in dynamical systems. In this paper, we give a further example of the interplay between these two fields in this family of maps.

It is known [DGH, BR] that there is an open set of λ values in the complex plane for which E_λ has a unique attracting fixed point. For these values the basin of attraction Ω_λ of the fixed point is a simply connected region which is completely invariant under E_λ. The Riemann Mapping Theorem, therefore, gives a uniformization ϕ_λ of Ω_λ, and we study the dynamics induced by ϕ_λ on the unit disk. The induced map assumes the form

$$T_\mu(z) = \exp\left(i\left(\frac{\mu + \bar{\mu}z}{1+z}\right)\right)$$

where the parameter μ lies in the upper half plane and depends on λ. T_μ extends analytically to the boundary of D, with the exception of the special point -1, and the map induced on the boundary is the well known Baker transformation.

On the other hand, E_λ extends continuously to $\partial\Omega_\lambda - \infty$, which is the Julia set of E_λ, and one may ask what type of correspondence ϕ_λ induces between ∂D and $\partial\Omega_\lambda$. We will describe this correspondence in detail, and show that *all* the radial limits of ϕ_λ exist.

This implies that ϕ_λ has a well-defined extension to the boundary of D. However, it is a consequence of Caratheodory's Theorem [P] that this extension is highly discontinuous, since the boundary of Ω_λ is locally connected only at ∞.

§1. Basins of attraction for the exponential map. Let $E_\lambda(z) = \lambda e^z$ where $\lambda \in \mathbb{C}$ and $\lambda \neq 0$. Let $C = \{\lambda \in \mathbb{C} | \lambda = \zeta e^{-\zeta}$ for some $\zeta \in \mathbb{C}$ with $|\zeta| < 1\}$, C consists of points in the interior of a cardioid in the λ-plane.

LEMMA 1.1. *E_λ has an attracting fixed point ω_λ if and only if $\lambda \in C$. Moreover, $\omega_\lambda = \zeta$, where $\lambda = \zeta e^{-\zeta}$.*

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