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CUBIC HYPERSURFACES AND A RESULT OF HERMITE

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Dedicated to Yu. I. Manin

Let k be an infinite field, and K/k a separable extension of degree 5. The following result is due to Hermite [5]: *there exists an element η in K whose minimal polynomial (over k) is of the form $x^5 + ax^3 + bx + c$.*

In his proof Hermite used a rather complicated invariant. Recently, Serre asked whether there is a more natural approach in terms of cubic surfaces. Indeed it is easy to reformulate Hermite's result as asserting the existence of a k-point on some specific cubic surface (1.8), which depends on the extension K/k.

It seems that this question originated from a letter of Joe Buhler to Serre (July 1982), in which Buhler mentioned Hermite's result and asked about possible generalizations to higher degrees.

In this paper we give a new proof of Hermite's result by studying the lines lying on the corresponding cubic surface. We also prove a generalization to degree 6 by constructing some curves in $\mathbb{P}^4$. Both techniques were already used in [2], though in simpler situations. The results of [2] also imply generalizations to degree 7 or 8, provided k belongs to some special class of fields, which includes all local fields.

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§1. Reformulation of Hermite's result and generalization. The main purpose of this section is to introduce some notation. The equivalence of ($\mathbb{H}$) and ($\mathbb{H}'$) is almost evident, but the argument supplied below (Proposition 1.1) gives us the opportunity to prove some auxiliary results, which will be needed later on.

Notation. If $K = k(\theta)/k$ is a separable extension of degree d, we denote by $\tau_i (i = 1, \dots, d)$ the various k-embeddings of K in a fixed algebraic closure $\bar{k}$ of k. We consider the formal expression

$$x = x_0 + x_1\theta + \cdots + x_{d-1}\theta^{d-1}, \tag{1.1}$$

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