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COMPARISON OF EQUIVARIANT ALGEBRAIC AND TOPOLOGICAL K-THEORY

R. W. THOMASON

Let k be a separably closed field, G a linear algebraic group scheme over k, and X a separated scheme of finite type over k. Fix a prime power ℓ^ν invertible in k. Suppose G acts on X. For the main result, assume that G is reduced.

The main result of this paper is that if the mod ℓ^ν equivariant algebraic K-groups are localized by inverting the Bott element, and then completed with respect to the augmentation ideal of the representation ring of G, then the results are isomorphic to the equivariant topological K-groups

$$G/\ell^\nu_*(G, X)[\beta^{-1}]^{\wedge}_{IG} \cong G/\ell^{\nu\,\mathrm{Top}}_*(G, X). \tag{0.1}$$

This result is for the K-theory of coherent G modules on X, $G_*(G, X)$, and the topological K-homology theory $G/\ell^{\nu\,\mathrm{Top}}_*(G, X)$.

For X and G smooth over k, the isomorphism also holds for the K-theory of algebraic G-vector bundles on X and equivariant topological K-cohomology.

$$K/\ell^\nu_*(G, X)[\beta^{-1}]^{\wedge}_{IG} \cong K/\ell^{\nu\,\mathrm{Top}}_*(G, X). \tag{0.2}$$

These theories were studied in [T2], [T3]. For $k = \mathbf{C}$, $K/\ell^{\nu\,\mathrm{Top}}_*(G, X)$ are the usual mod ℓ^ν topological K-cohomology groups of the space $EG \times^G X$. $G/\ell^{\nu\,\mathrm{Top}}(G, X)$ is the dual Borel equivariant homology theory with locally compact supports. The result is another example where inverting the Bott element transforms algebraic geometry into topology.

Some extensions of the result are possible. The separably closed base field k may be replaced by a ring of integers in a number field localized away from ℓ, or by $\hat{\mathbf{Z}}_p$ for $p \neq \ell$, or by $\hat{\mathbf{Q}}_p$, or by a finite field, provided that k contains $\sqrt{-1}$ if $\ell = 2$. The group G over k may be replaced by a smooth over S, closed subgroup-scheme of GL_n over S, where S is separated and of finite type over k: i.e. one may study continuous families of linear algebraic groups.

However, I doubt that the result remains true if X is not of finite type over k, or if k is a field with infinite etale cohomology groups, e.g. if k is a number field. This is because the statements (0.1) and (0.2) are expressed on the superficial level of homotopy groups, rather than on the deeper level of spectra, and so depend on the IG-adic completion process being exact on the level of homotopy groups. This necessitates strong finiteness assumptions, which can be met only

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