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GENERA OF THE ALTERNATING LINKS

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In 1958 Crowell [C] and Murasugi [M] independently showed that by applying Seifert's algorithm to an alternating presentation of an oriented link one obtains a surface of minimal genus. Their method for such a knot K was to show that Seifert's inequality

$$\text{genus } K \geqslant \tfrac{1}{2} \text{ degree Alexander polynomial of } K$$

was in fact equality. By suitably generalizing the above inequality they obtain the result for oriented links. Both proofs involve rather complicated combinatorial arguments.

In contrast to the algebraic approach it was shown using foliations in [G1] that these surfaces are minimal genus. That proof relies on a delicate construction and a result of Thurston [T] which relates minimal genus surfaces to foliations.

In this paper we give a completely elementary and self contained proof that the surfaces obtained by applying Seifert's algorithm to alternating projections of oriented links are minimal genus.

Notation 1. If S is a compact oriented n-manifold than ∂S denotes its oriented boundary. If S is properly embedded in the manifold M then $[S]$ denotes the homology class represented by S as an element of $H_n(M, \partial M)$. $\mathring{E}$ denotes the interior of E and $N(E)$ denotes a tubular neighborhood of E.

Definition 2. If L is an *oriented* link in S^3 then a *Seifert surface* for L is an *oriented* surface R embedded in S^3 such that $\partial R = L$ and no component is closed. A *surface of minimal genus* for L is a Seifert surface S which satisfies $\chi(S) \geqslant \chi(T)$ for T any other Seifert surface. Note that the Euler characteristic of a minimal genus surface depends on how the link is oriented.

LEMMA 3. *If L is an oriented link in S^3 and S is a Seifert surface for L which is not minimal genus, then there exists a Seifert surface T for L such that $\chi(T) > \chi(S)$ and $\mathring{T} \cap \mathring{S} = \varnothing$.*

Proof. (This is essentially lemma 3.5 of [G2].) Let T be a Seifert surface for L such that $\chi(T) > \chi(S)$. View S and T as properly embedded surfaces in $S^3 - \mathring{N}(L)$. As elements of $H_2(S^3 - \mathring{N}(L), \partial N(L))$, $[S] = [T]$ since such elements are determined by their intersection numbers with meridians of $N(L)$. It

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