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# DEGENERATIONS OF RATIONAL SURFACES WITH TRIVIAL MONODROMY

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One of the most basic results in the theory of ruled surfaces is that any degeneration of rational curves may be smoothly blown down to a trivial degeneration. To study threefolds fibered by rational surfaces, one is therefore led to ask whether degenerations of rational surfaces may also be smoothly blown down to a trivial degeneration or at least dominate a trivial degeneration. The principal result of this paper is that any monoidally Kähler (Kähler after smooth blow-ups and blow-downs) degeneration of rational surfaces with trivial monodromy may be modified by smooth blow-ups and blow-downs (in an explicit manner) to a degeneration of the projective plane which dominates the trivial degeneration. Note that any degeneration of rational surfaces has finite monodromy and so, after base-change by a finite map, the pullback is a degeneration with trivial monodromy.

In Section One we birationally modify any monoidally Kähler degeneration of rational surfaces with trivial monodromy to a monoidally Kähler degeneration of projective planes (Theorem 1.5). If $X \to \Delta$ is the degeneration, with general fiber $X_t$, we first identify $H^2(X; \mathsf{Z})$ with $\mathrm{Pic}_h X$, the holomorphic Picard group of $X$ (Lemma 1.2) and use the Clemens–Schmid exact sequence to show that any line bundle on $X_t$ may be extended to a line bundle on $X$ (Lemma 1.3). Then we show that any section of a line bundle $L_t$ on $X_t$, where $H^1(X_t, L_t) = 0$, comes from a section of a line bundle on $X$ extending $L_t$ (Lemma 1.4). We use lemmas 1.3 and 1.4 to realize an exceptional curve on $X_t$ as the restriction of a generically contractible ruled surface in $X$. After blow-ups and blow-downs, this surface may be smoothly contracted. Thus we are reduced to considering degenerations of $\mathsf{F}_N$, $N \neq 1$. Finally, we blow up certain sections of $X \to \Delta$ to enable us to realize elementary modifications $\mathsf{F}_N \dashrightarrow \mathsf{F}_{N-1}$ in the total space. These techniques also allow us to modify a degeneration of Enriques' surfaces to a degeneration of minimal Enriques' surfaces.

In Section Two, we prove that if $X \to \Delta$ is a degeneration of projective planes, then $X$ dominates the trivial degeneration if and only if $X$ contains a smooth surface $H$, ruled over $\Delta$, such that $H \cdot X_t$ is the class of a line in $X_t$, and if $\mathscr{H}$ is a curve in $H \cap X_0$, then $\mathscr{H} \cdot H \geqslant 0$ (Theorem 2.5). There are examples of $X$ with $H$ satisfying all of these conditions except the last one (see 4.4). We show that if $X \to \Delta$ is a monoidally Kähler degeneration of projective planes, then $X$ may be

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