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THE POINCARÉ INEQUALITY FOR VECTOR FIELDS SATISFYING HÖRMANDER'S CONDITION

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§1. Introduction. Our purpose in this paper is to prove a Poincaré-type inequality of the form

$$(*) \qquad \int_{B(r)} |f - f_{B(r)}|^2 \leqslant Cr^2 \int_{B(r)} \sum_{i=1}^{m} |X_i f|^2, \qquad \text{for all } f \in C^\infty\left(\overline{B(r)}\right)$$

where $X_1, \ldots, X_m$ denote vector fields on $\mathbf{R}^d$ satisfying Hörmander's condition, $B(r)$ denotes a ball of radius r with respect to a natural metric associated with $X_1, \ldots, X_m$, and $f_{B(r)}$ denotes the average value of f on the ball $B(r)$. This inequality is the same as finding a lower bound $1/Cr^2$ on the least nonzero eigenvalue in the Neumann problem for $L = \sum_{i=1}^m X_i^* X_i$ on $B(r)$.

The operator L has been the subject of many investigations since Hörmander's proof of subellipticity [9, 14]. Together they yield a very complete description of solutions to L including a strict maximum principle, function space estimates, the size of the fundamental solution, and asymptotics of the eigenvalues [2, 3, 4, 5, 6, 17, 18, 19, 20]. The entire picture is governed by the geometry of the balls $B(r)$. Our problem is different from most of those previously studied because rather than being purely local, it involves a global consideration, namely the character of the boundary of $B(r)$. As we shall see in the final section, it is easy to give examples of smooth domains for which our Poincaré inequality fails.

In [2] J.-M. Bony deduced a qualitative Harnack inequality for L from his strict maximum principle and asked whether a direct proof was possible. Inequality (*) provides such a proof: it is central to a proof of a quantitative, scale-invariant form of Harnack's inequality for L along the lines of Moser's proof of the Harnack inequality for uniformly elliptic divergence-form operators with bounded measurable coefficients [16]. The analogy with uniformly elliptic operators is even more striking when one considers the more general class of Fefferman-Phong type operators [3]. This class allows for the possibility that the quadratic form associated to L cannot be expressed smoothly as a sum of squares. The condition necessary and sufficient for subellipticity is that the ball $B(r)$ contain a Euclidean ball of radius r^N. In a forthcoming article Antonio Sánchez-Calle and the author will establish a universal Poincaré-type inequality depending only on N and bounds in the C^k norm on the coefficients of L. The

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