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ZARISKI DECOMPOSITION OF DIVISORS ON ALGEBRAIC VARIETIES

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Introduction. In his article [Z] Zariski solved the Riemann–Roch problem for high multiples of an effective divisor on a smooth projective surface. Zariski needed a decomposition of an effective divisor D into a numerically effective Q-divisor Δ, and an effective Q-divisor F such that $h^0(n\Delta) = h^0(nD)$ whenever n is a positive integer for which $n\Delta$ is an integral divisor. Such a decomposition is called a Zariski decomposition.

In this paper the question of when an effective divisor on a smooth projective r-dimensional variety has a Zariski decomposition is investigated. The main result is an example of a divisor D on a 3-fold V with D-dimension $\kappa(D, V) = 3$, and such that for any birational morphism $f\colon V' \to V$ the pullback $f^*(D)$ does not have a Zariski decomposition. This provides a counter example to a conjecture of Fujita given in [F2]. However, we show that divisors with D-dimension 1 or 2 always have a Zariski decomposition after taking their pullback by a suitable birational morphism.

The main result of section 2 is a proof that the canonical ring of a 3-fold V of general type is finitely generated if there is a canonical divisor K on V, and a birational morphism $f\colon V' \to V$ such that $f^*(K)$ has a Zariski decomposition. This was essentially proven by Benveniste in [B], but we were able to show that some restrictions Benveniste placed on the intersection theory of the prime components of K are always satisfied by a Zariski decomposition.

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Notations. All varieties are assumed to be projective and to be defined over C.

Let V be a smooth variety. Let Ω, Δ be irreducible subvarieties. Define:

$$\operatorname{ord}_\Delta(\Omega) = \begin{cases} 0 & \text{if } \Delta \not\subset \Omega \\ r & \text{if } \Delta \text{ has multiplicity } r \text{ in } \Omega. \end{cases}$$

Let D be a divisor on V. $\kappa(D, V)$ is the D-dimension of D, as defined by Iitaka [I]. $\sim$ denotes linear equivalence. $\approxeq$ denotes numeric equivalence. Z-divisors,

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