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ON A REMARK OF O'NEILL

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§0. Introduction. Let $S^n(r)$ denote the round sphere of radius r in $\mathbf{R}^{n+1}$ and S^n denote $S^n(1)$. Let $\pi : S^n \to B$ be a Riemannian submersion and $\beta : \mathbf{R} \to B$ a unit speed geodesic. Let T and A be the structural tensors of π as introduced in [1] by O'Neill. In [2] O'Neill defines the norm $\alpha(\beta)$ of A along the geodesic β to be the maximum of $|A_x y|$ for x, y unit orthonormal vectors tangent to B at a point of β with the additional condition that one of them (say x for definiteness) be tangential to β.

Towards the end he remarks (in [2]) that if for every geodesic β, $\alpha(\beta) \leqslant 1$, then "the geodesics of B have essentially the same conjugacy properties as for $\mathbf{C}P^n$ or QP^n." In this note we wish to observe that

(1) $\alpha(\beta) \leqslant 1$ implies that the fibres of π are totally geodesic so that B is in fact isometric to one of the spaces $\mathbf{C}P^n, \mathbf{Q}P^n$ or $S^8(\frac{1}{2})$. (See [3].)

(2) It is enough to start with a Riemannian foliation of S^n i.e., a foliation with "bundle-like" metric and assume that $\alpha(\beta) \leqslant 1$ for horizontal geodesics to conclude that the leaves are totally geodesic and hence we do have a global Riemannian submersion of S^n.

(3) Hence if we start with a Riemannian foliation which is not obtainable by a global Riemannian submersion, then $\alpha(\beta) \leqslant 1$ will not be satisfied for all horizontal geodesics β and such foliations of course exist. Take for example a nowhere vanishing Killing vector field on S^{2n+1} which admits noncompact orbits.

§1. Equation {2} of O'Neill. Let $\pi : M \to B$ be a Riemannian submersion and T and A its structural tensors. Then equation $\{2\}$ of O'Neill as given in [1] states that

$$\langle R_{XV}Y, W\rangle = \langle(\nabla_X T)_V W, Y\rangle + \langle(\nabla_V A)_X Y, W\rangle$$
$$- \langle T_V X, T_W Y\rangle + \langle A_X V, A_Y W\rangle$$

where X, Y are horizontal and V, W the vertical vector fields. Let β be a horizontal geodesic and $\{V_i\}_{i=1}^k$ a parallel orthonormal frame of vertical fields along β (parallel under $\hat{\nabla}$, the induced connection on the verticle subbundle). Put $\beta' = X$.

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