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CLOSED FAMILIES OF SMOOTH SPACE CURVES

MEI-CHU CHANG AND ZIV RAN

Consider algebraic families of nonsingular curves in a fixed projective space. Common experience would suggest that "most" nontrivial such families possess singular limits, i.e., their parameter spaces must be *incomplete*.

In this paper we focus attention on the case where this does not happen, i.e., on families $\{Y_\lambda : \lambda \in \Lambda\}$ of nonsingular curves in $\mathbf{P}^n$ whose parameter space Λ is itself complete (such families will be called *closed* families). Our main result (Theorem 2.1) gives a bound on the dimension of such Λ: namely, if the general Y_λ is nondegenerate and $Y_\lambda \neq Y_{\lambda'}$ if $\lambda \neq \lambda'$, then $\dim \Lambda \leqslant n - 1$; in fact our result is slightly more general and allows $\bigcup_\lambda \operatorname{sing}(Y_\lambda)$ to be finite, provided a certain separability condition is satisfied (which is automatic in char $\cdot$ 0). An easy construction (Example 2.2) yields plenty of families $\{Y_\lambda\}$ with $\bigcup \operatorname{sing}(Y_\lambda)$ finite (respectively, empty) and $\dim \Lambda = n - 2$ (resp. $n - 3$), so our bound is not very far from being sharp. In addition, we will give some general bounds for closed families of nonsingular subvarieties $\{Y_\lambda : \lambda \in \Lambda\}$ of a fixed ambient variety X: namely we will show that in general, $\dim \Lambda \leqslant (\dim(Y_\lambda) + 1) \cdot \operatorname{codim}(Y_\lambda, X)$, while if all the Y pass through a fixed point, then $\dim \Lambda \leqslant \dim(Y_\lambda)\operatorname{codim}(Y_\lambda, X)$. As the case of linear subspaces of $\mathbf{P}^n$ shows, these bounds are sharp. Also, we prove the ampleness of a certain line bundle associated to a closed family of nonsingular subvarieties of $\mathbf{P}^n$ or an abelian variety, somewhat analogous to a theorem of Arakelov for abstract families of curves.

Our main result may be compared to a recent theorem of S. Diaz which shows that for a nondegenerate abstract family $\{Y_\lambda : \lambda \in \Lambda\}$ of nonsingular curves of genus g one has $\dim \Lambda \leqslant g - 2$. While Diaz's bound does imply some bounds for families of space curves, such bounds are, in general, much weaker than ours.

The contents of this paper are as follows. In §1 we begin by analyzing the infinitesimal situation corresponding to a closed family of nonsingular subvarieties, say curves, on a variety, all passing through a fixed point P. This leads us to consider closed families of non-reduced "curve-like" (i.e. curvilinear) schemes supported at P. We show mainly that there are no nontrivial such families with fixed Zariski tangent space. This yields our general bounds for families of nonsingular subvarieties, and the ampleness result mentioned above. In §2 we prove our main bound by combining the estimates of §1 with projective techniques.

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