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LEBRUN'S NONREALIZABILITY THEOREM IN HIGHER DIMENSIONS

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1. Introduction. In this article we generalize the realizability criterion of LeBrun [5] to CR twistor manifolds corresponding to higher dimensional Riemannian manifolds. Associated to every Riemannian manifold (M, g) there is a CR manifold (N, D), where D is the distribution defining the $\bar{\partial}_b$ equations of the CR structure. If M is of dimension 3, N has the structure of a CR hypersurface, but in higher dimensions this is no longer true. In fact, if $\dim_R M = n$, we have $\dim_R N = 3n - 4$ and $\dim_C D = n - 1$. Thus, if the CR structure is induced by an embedding of N in $\mathfrak{C}^d$, the best we could have is $d = 2n - 3$ and N realized as a submanifold of codimension $n - 2$. Nevertheless the realizability criterion of LeBrun still holds: N is everywhere locally realizable if and only if (M, g) is conformal to a real-analytic Riemannian manifold. In fact, the construction is, in some sense, easier if $n > 3$.

There is a fibration $\pi : N \to M$ so that the fibers are compact complex submanifolds of N isomorphic to quadrics in $\mathfrak{P}^{n-1}$. If $\mathfrak{Q}$ is the space of quadrics, the metric thus defines a smooth map $g : M \to \mathfrak{Q}$. Under the hypothesis of realizability we can, using the criterion of Boggess and Polking [1], embed N as a closed submanifold of a complex manifold T with $\dim T = 2n - 3$, so that CR functions on N extend to holomorphic functions on T. Let S be the space of compact complex submanifolds of T; S carries a natural complex structure [6]. A cohomological calculation along the fibers of $\pi^{-1}(0)$ verifies the condition [4] for $\{\pi^{-1}(0)\}$ to be a regular point of S. Then there is a natural smooth embedding of M into S: $\phi(m) = \{\pi^{-1}(m)\}$. Now, following LeBrun's argument, if $n > 3$, S is a complexification of M, and there is a holomorphic map $\hat{g} : S \to \mathfrak{Q}$ which extends the "metric map" $g : M \to \mathfrak{Q}$. This proves the real-analyticity.

The aim of this work was to extend LeBrun's ideas to produce a nondegenerate CR hypersurface which is not realizable of dimension greater than 5. Since [3] has examples of Levi signature $(n, 1)$, the task is to find such an example of Levi signature (p, q), $p \geqslant q \geqslant 2$. We have not found an example; the best we can do is a hypersurface whose Levi form has signature $(2, 2)$, but has one zero eigenvalue.

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