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CONVEXITY OF SOLUTIONS OF SEMILINEAR ELLIPTIC EQUATIONS

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Introduction. Consider an elliptic problem

$$\begin{aligned} \Delta u &= f(u) \qquad \text{in } \Omega, \\ u &= M \qquad \text{on } \partial\Omega \end{aligned} \tag{1}$$

where Ω is a convex domain in $\mathbf{R}^2$ and M is either a real constant or $+\infty$. We are interested in establishing, for a suitable monotone function $g(t)$, that for any solution u of (1)

$$g(u) \text{ is strictly convex in } \Omega. \tag{2}$$

Some results of this type are well known. Thus in the case of the principal eigenfunction:

$$\begin{aligned} \Delta u + \lambda u = 0, \qquad u > 0 \qquad &\text{in } \Omega, \\ u = 0 \qquad &\text{on } \partial\Omega \end{aligned} \tag{3}$$

it was first proved by Brascamp and Lieb [5], and more recently also by Caffarelli and Spruck [6] and Korevaar [14], that $-\log u$ is a convex function (even when Ω is n-dimensional); the convexity of the level lines was also proved by Acker, Payne and Philippin [1]. In case $M > 0$ and under suitable assumptions in f (in (1)), for instance, if $f(u) = (u^+)^p f_0(u)$, $0 < p < 1$, $f_0 \in C^2$ and f monotone increasing, it was proved by Friedman and Phillips [8] that the set $\{u = 0\}$ is convex.

Our method is based on first working with an elliptic equation of the form

$$\Delta w = G(w) + H(w)|\nabla w|^2 \tag{4}$$

and establishing (under suitable assumptions on G and H) the following result (in §1):

If w is convex in Ω and if Ω does not coincide with an infinite strip, then w is strictly convex in Ω. (5)

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