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GEOMETRY OF A CATEGORY OF COMPLEXES AND ALGEBRAIC K-THEORY

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Introduction. It is well known (cf. [8]) that the group K_0 of an abelian category $\mathscr{A}$ can be described in terms of its derived category $D^b(\mathscr{A})$: $K_0(\mathscr{A})$ has the presentation by generators $[X]$, $X \in \mathrm{Ob}(\mathscr{A})$, and relations $[X]+[Y] = [Z]$ for X, Y, Z being included in an exact triangle

$$X \to Y \to Z \to X[1].$$

The present notes arose from an attempt to invent an analogous description for higher K-groups $K_i(\mathscr{A})$, $i > 0$.

For example, one could expect that in the definition of $K_1(\mathscr{A})$ would take part something like "exact triangles factored by exact octahedrons", etc. We show below (cf. Theorem 3.4) that in a sense this is really so.

Let us describe briefly the idea of our construction. One knows that the group K_i of a monoidal category M is the $(i+1)$-th homotopy group of its "classifying space" BM (cf. [7]). One can imagine a monoidal category as "a group up to a canonical isomorphism", and its classifying space as an analogue of the classifying space of a group. Recall that for a group A the latter space is the simplicial set, whose set of n-simplices, $(BA)_n$, is just A^n, and the faces are defined by the formula

$$d_i(a_1, \ldots, a_n) = \begin{cases} (a_2, \ldots, a_n) & \text{for } i = 0 \\ (a_1, \ldots, a_i + a_{i+1}, \ldots, a_n) & \text{for } i = 1, \ldots, n-1 \\ (a_1, \ldots, a_{n-1}) & \text{for } i = n. \end{cases}$$

Similarly, the operation of cone in the category of complexes $C^b(\mathscr{A})$ is an analogue of the "division" $(x, y) \to (x^{-1}y)$ in a group A.[1] Further, note that one can describe the faces of BA in other co-ordinates, in terms of division. Namely, one can put $B'A = A^n$,

$$d_i(a_1, \ldots, a_n) = (a_1, \ldots, \hat{a}_i, \ldots, a_n) \quad \text{for } i \geqslant 1 \text{ and}$$

$$d_0(a_1, \ldots, a_n) = (a_1^{-1}a_2, \ldots, a_1^{-1}a_n),$$

and one has an isomorphism $B'A \cong BA$.

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[1] The role of the "associativity isomorphism" is played in $C^b(\mathscr{A})$ by the quasi-isomorphism of octahedron (see (1.1)).