HYPERELLIPTIC QUASI-PERIODIC AND SOLITON SOLUTIONS OF THE NONLINEAR SCHRÖDINGER EQUATION

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In physical applications, both forms of the nonlinear Schrödinger equation are of interest:

$$\text{“nonfocussing”}, \qquad u_{xx} - \sqrt{-1}\,u_t = 2|u|^2 u \qquad (\text{NLS})_1$$

$$\text{“focussing”}, \qquad u_{xx} - \sqrt{-1}\,u_t = -2|u|^2 u. \qquad (\text{NLS})_2$$

The qualitative behavior of the solutions changes substantially with the sign.

In finding solutions we shall go through an intermediate step, a complexified version of the equations:

$$\begin{cases} u_{xx} - \sqrt{-1}\,u_t = -2u^2 v \\ v_{xx} + \sqrt{-1}\,v_t = -2v^2 u \end{cases} \qquad (\text{NLS})$$

where u, v are holomorphic (for small $|x|, |t|$) functions of the complex variables x, t; (NLS) specializes to $(\text{NLS})_1$, $(\text{NLS})_2$ resp. when the condition $v = \mp \bar{u}$ is satisfied; this requires finding reality conditions on (x, t). After the discovery by Zakharov and Shabat ([31]) that NLS can be represented as a “Lax pair”, $L_t = [B, L]$, numerous constructions that had been developed for solving the analogous Lax pair of the Korteweg–de Vries (KdV) equation were successfully applied to NLS. The main features to be investigated, typically on a space of functions of x with prescribed boundary conditions at $\pm\infty$, were: (i) inverse scattering problem ([2], [15], [31], [32]); (ii) direct method, τ function and solitons ([9], [10], [13]); (iii) Hamiltonian formalism and complete integrability ([29]); (iv) integrability as a continuum of harmonic oscillators ([4]); on a space of periodic or quasi-periodic functions, the questions were: (i)′ construction of solutions by the use of algebraic geometry, specifically hyperelliptic theta functions ([3], [11], [12], [14]) and initial-value problem for periodic potentials ([1]).

The contribution of the present paper is on one hand a more complete description of the structure discussed in (i)′, on the other hand the disclosure of

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