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BLOWINGS-UP OF $\mathbf{P}^2$ AND THEIR BLOWINGS-DOWN

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Consider a smooth projective surface S, over an algebraically closed field k of arbitrary characteristic, such that there is a proper birational morphism $S \to \mathbf{P}^2$; i.e., S is obtained by blowing-up a finite sequence of points, possibly infinitely near, of $\mathbf{P}^2$ [5, V.5.4]. It often happens that there is more than one such blowing-down $S \to \mathbf{P}^2$ of S, even if we do not regard as essentially distinct those morphisms differing only by a projective linear transformation of $\mathbf{P}^2$. That is, S may dominate a nontrivial birational transformation of $\mathbf{P}^2$. In this paper we characterize the blowings-down of S and, under the condition that S have an irreducible and reduced anticanonical divisor, we give criteria that decide whether there are infinitely many or only finitely many blowings-down.

Our approach is to factor such a morphism $S \to \mathbf{P}^2$ into a sequence $S = S_n \to S_{n-1} \to \cdots \to S_0 = \mathbf{P}^2$ of blowings-up $\pi_i : S_i \to S_{i-1}$ of S_{i-1} at a point p_i of S_{i-1} [5, V.5]. Then $S \to \mathbf{P}^2$ determines the classes $\mathcal{E}_0, \ldots, \mathcal{E}_n \in \operatorname{Pic} S$ of, respectively, the total transforms of a line in $\mathbf{P}^2$ and the points $p_1, \ldots, p_n$. The elements $\mathcal{E}_0, \ldots, \mathcal{E}_n$ form a free $\mathbf{Z}$-module basis of $\operatorname{Pic} S$ and determine $S \to \mathbf{P}^2$ up to a projective linear transformation of $\mathbf{P}^2$ [5, V]. Modifying slightly the usage of [8], we call such a collection $\mathcal{E} = \{\mathcal{E}_0, \ldots, \mathcal{E}_n\}$ of classes an *exceptional configuration*.

If S is a surface with an exceptional configuration $\mathcal{E} = \{\mathcal{E}_0, \ldots, \mathcal{E}_n\}$, Nagata [13] defines an action by a group W on the group $\operatorname{Pic} S$ of divisor classes of S. He essentially shows (cf. (0.1)) that every exceptional configuration of S is a W-translation of $\mathcal{E}$ and, if S is a blowing-up of $\mathbf{P}^2$ at sufficiently general points, that every W-translation of $\mathcal{E}$ is an exceptional configuration of S. DuVal [2] had already identified W as a Coxeter group; indeed, W is the Weyl group of a root system embedded in $\operatorname{Pic} S$, and Manin [11], Demazure [1], and Looijenga [8] make use of this root system in studying blowings-up of $\mathbf{P}^2$. In terms of this root system, our result (1.1) of section 1 refines the result of Nagata: for an element w of W, $w\mathcal{E}$ is an exceptional configuration of S if and only if $w\mathcal{E}_0$ is numerically effective and $w^{-1}r$ is a positive root whenever r is an effective root of the root system.

Root system techniques seem especially appropriate for studying a surface V with an exceptional configuration, when the anticanonical class of V has a reduced and irreducible section. Looijenga [8] shows that the set of nodal (i.e., effective and irreducible real) roots of V determines the exceptional configura-

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