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GROUP COMPLETIONS AND FURSTENBERG BOUNDARIES: RANK ONE

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For every noncompact, connected, semisimple Lie group G with finite center there is an associated compact space $B(G)$, the Furstenberg maximal boundary of G, on which G acts. $B(G)$ is important for the theory of harmonic functions on G [2], and is a crucial ingredient of Mostow's proof of rigidity for uniform lattices of semi-simple Lie groups [4]. In 1970 Furstenberg [3] asked whether one could construct $B(G)$ directly from a lattice H in G, and he gave there a few attempts in terms of probability spaces.

In [1] I gave a construction which, given a finitely generated group Λ (and a finite generating set for Λ), produces a compact metric space $\overline{\Lambda}$ on which Λ acts. If Λ is a lattice in $\mathrm{PSL}(2,\mathbf{C})$, then there is a Λ-equivariant homeomorphism $\phi:\overline{\Lambda}\to B(\mathrm{PSL}(2,\mathbf{C}))$, and if Λ is a uniform lattice of $\mathrm{SO}_0(1,n)$ then there is a Λ-equivariant homeomorphism $\phi:\overline{\Lambda}\to B(\mathrm{SO}_0(1,n))$. I was not aware at the time of Furstenberg's larger question, and was working instead from an interest in real hyperbolic space.

In this paper we show that if G is a noncompact, connected semisimple Lie group with finite center and $\mathbf{R}$-rank one and Λ is a uniform lattice of G (that is, a discrete subgroup of G so that G/Λ is compact), then there is a Λ-equivariant homeomorphism $\phi:\overline{\Lambda}\to B(G)$. The proof follows the main proof in [1], and uses the classification of noncompact, Riemannian symmetric spaces with $\mathbf{R}$-rank one and negative curvature. In §1 we recall the construction of $\overline{\Lambda}$ and some properties of it. In §2 we give models for the noncompact, Riemannian symmetric spaces with $\mathbf{R}$-rank one and negative curvature. In §3 we give the proof and point out that this construction cannot always work if G has rank greater than one.

§1. Group completions. In this section we give the definition of $\overline{\Lambda}$ and recall some basic facts about $\overline{\Lambda}$. For more information, the reader is urged to consult [1]. Let Λ be a finitely generated group, and let Σ be a finite generating set for Λ which does not contain the identity. The *graph* of Λ (with respect to Σ) is the 1-complex $K(\Lambda,\Sigma)$ whose vertices are the elements of Λ and whose edges correspond to unordered pairs $a,b\in\Lambda$ such that $a^{-1}b\in\Sigma$ or $b^{-1}a\in\Sigma$. Define a norm on Λ by setting $|a|$ to be the minimal word length of a in Σ, and enlarge this to a left-invariant metric, called the word metric, on Λ by $(a,b)=|a^{-1}b|$. Let

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