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LOCALLY COERCIVE NONLINEAR EQUATIONS, WITH APPLICATIONS TO SOME PERIODIC SOLUTIONS

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1. Introduction. Many problems in analysis reduce to solving an equation of the form

$$Au = f, \tag{NL}$$

where A is an operator on a space into another space, u is the unknown, and f is a given element. In this paper we assume that A is a map on a subset $D(A)$ of a Banach space Y into another Banach space Y^*. (Y^* need not be the dual of Y in the usual sense.) In an ideal situation, (NL) will have a solution u for every $f \in Y^*$. There is a large literature on the "surjectivity" of this kind, including those related to monotone operators and their generalizations (see, for example, Browder [1]).

In the present paper we want to generalize the problem and seek sufficient conditions for (NL) to have a solution u for all *sufficiently small* f, in a sense to be specified below.

At the same time, we find it convenient to modify the setting of the problem by assuming that A is an operator mapping Y into a Banach space V^* containing Y^* as a subset, although the right member f of (NL) is in Y^*. This amounts to extending A from the original domain $D(A)$ to all of Y by admitting ideal elements as its values. It is a convenient means of handling the operator A when its domain $D(A)$ is not easy to describe explicitly. For example, let A be a nonelliptic system of first-order differential operators, considered an unbounded closed linear operator in $L^2(\mathsf{R}^m)$. Its domain is not easily described. But if we allow A to act on all of L^2 with values in H^{-1}, and solve (NL) with $f \in L^2$, the solution u will be in $D(A)$ automatically. In fact this is exactly what one does to define $D(A)$ in a "weak" sense. Actually we need not define A on *all* of Y; it would suffice to define it on a ball of Y, for example.

Thus the setting of our problem will be the following.

(i) $\{Y, Y^*\}$ is a pair of real Banach spaces *in duality*. This means that there is a nondegenerate continuous bilinear form $\langle\ ,\ \rangle$ on $Y \times Y^*$. Moreover Y is reflexive and separable. (We do not assume that $|\langle y, f\rangle| \leqslant \|y\|_Y \|f\|_{Y^*}$; if it is satisfied, we say $\{Y, Y^*\}$ is in *metric* duality.)

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