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INFINITESIMAL RIGIDITY OF $S^1 \times \mathsf{RP}^n$

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Let (X, g) be a compact Riemannian symmetric space. We say that a symmetric 2-form h on X satisfies the zero-energy condition if for all closed simple geodesics γ of X the integral

$$\int_\gamma h = \int_0^L h(\dot\gamma(s), \dot\gamma(s))\, ds$$

of h over γ vanishes, where $\dot\gamma(s)$ is the tangent vector to the geodesic γ parametrized by its arc-length s and L is the length of γ. We recall that the Lie derivative of g along a vector field on X always satisfies the zero-energy condition. Let g_t be a deformation of the metric $g_0 = g$. If the spectrum of the lengths of the closed geodesics of g_t is independent of the parameter t, then the infinitesimal deformation $h = (d/dt)g_t|_{t=0}$ is a symmetric 2-form satisfying the zero-energy condition. An infinitesimal rigidity problem for g is to determine whether the only symmetric 2-forms on X satisfying the zero-energy condition are the Lie derivatives of the metric g. If X is of rank one and is not a sphere, this is the infinitesimal version of the Blaschke conjecture: a metric on X, all of whose geodesics are closed and of length π, is isometric to the standard metric on X. For these spaces, the infinitesimal Blaschke problem has been answered affirmatively, namely by Michel [7] for RP^n, and by Michel [7] and Tsukamoto [10] for the other projective spaces (see also [2], [5]). Moreover, Michel [8] solved it for flat n-tori T^n. In this paper, we give a positive solution to this question for $S^1 \times \mathsf{RP}^n$, thus extending Michel's results on RP^n and on T^2. This is the only known example of a non-flat symmetric space of rank 2 whose metric is infinitesimally rigid.

In [4], we introduced an explicit resolution of the sheaf of Killing vector fields on X, which is an elliptic complex. The compatibility condition for the Killing operator is expressed by a third-order differential operator Q_g, which involves the infinitesimal orbit $\tilde{G}$ of the curvature under the action of the orthogonal group. We characterize the conformally flat symmetric Riemannian manifolds of dimension $\geqslant 4$ in terms of this orbit $\tilde{G}$ (Theorem 2.3), and compute $\tilde{G}$ in the case of a conformally flat manifold which is a product.

In our proof of the infinitesimal rigidity of $S^1 \times \mathsf{RP}^n$, we exploit the fact that this space is conformally flat and we use the above-mentioned geometric description of the elements of $\tilde{G}$. Let h be a symmetric 2-form on $X = S^1 \times \mathsf{RP}^n$, with $n \geqslant 2$, satisfying the zero-energy condition. By means of Michel's theorem

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