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CHARACTERIZATION OF H^1 BY SINGULAR INTEGRALS: NECESSARY CONDITIONS

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1. Introduction. On $\mathbf{R}^n$ it has been completely determined by Uchiyama [9] which finite collections $\{K_j\}$ of singular integral operators characterize the Hardy space H^1. That is, $f \in H^1$ if and only if each $K_j f \in L^1$ (in the sense of tempered distributions), and

$$C^{-1}\|f\|_{H^1} \leqslant \sum \|K_j f\|_{L^1} \leqslant C\|f\|_{H^1}.$$

In Christ and Geller [2] the analogous question is studied on the Heisenberg group $\mathbf{H}^n$, and a condition on the operators K_j sufficient to ensure that they characterize H^1 is given. Here that condition will also be shown to be necessary. The proof is not a straightforward adaptation of a familiar argument in Euclidean space. Instead, the problem is intimately related to the existence of singular pluriharmonic measures, and hence of inner functions, on the ball in $\mathbf{C}^{n+1}$. The idea of using singular measures for counter-examples in this context may also be found in Gandulfo, Garcia-Cuerva and Taibleson [5].

This note is an addendum to the work of Christ and Geller [2], to which the reader is referred for almost all notation and definitions. In particular, Hardy spaces on $\mathbf{H}^n$ are defined in terms of its group translation and dilation structures, rather than as on Euclidean space. A real singular integral operator is of the form

$$Kf(x) = \alpha f(x) + pv\int f(y)k(y^{-1}x)\,dy,$$

where $\alpha \in \mathbf{R}$, and k is real-valued, smooth away from the origin and has appropriate cancellation and homogeneity properties.

Recall that $\mathbf{H}^n$ has a one-parameter family of irreducible unitary representations $\{\pi_\lambda : \lambda \in \mathbf{R}\backslash\{0\}\}$ on a fixed infinite-dimensional Hilbert space $\mathscr{H}$. $\mathscr{S}(\mathscr{H})$ will denote the set of smooth vectors in $\mathscr{H}$: $v \in \mathscr{S}(\mathscr{H})$ if the map from $\mathbf{H}^n$ to $\mathscr{H}$ given by $x \to \pi_\lambda(x)v$ is C^∞. $\mathscr{S}(\mathscr{H})$ is independent of λ. In fact there is a standard orthonormal basis $\{e_j : j \in \mathbf{N}^n\}$, where $\mathbf{N} = \{0, 1, 2, \dots\}$ such that $v = \sum \alpha_j e_j$ is in $\mathscr{S}(\mathscr{H})$ if and only if $\sum |\alpha_j|^2 |j|^M < \infty$, for all $M < \infty$. If $f \in L^1(\mathbf{H}^n)$, then for each λ there is associated to f a bounded operator $\pi_\lambda(f)$ on $\mathscr{H}$, defined by $\pi_\lambda(f) = \int f(x)\pi_\lambda(x)\,dx$. There exists a unitary involution $\mathscr{J}$ on $\mathscr{H}$ such that $f \in L^1(\mathbf{H}^n)$ is real-valued if and only if $\pi_{-\lambda}(f) = \mathscr{J}\pi_\lambda(f)\mathscr{J}$, for all $0 \neq \lambda \in \mathbf{R}$. For any singular integral operator K there exists for each λ a bounded

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