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SINGULAR INTEGRAL CHARACTERIZATIONS OF HARDY SPACES ON HOMOGENEOUS GROUPS

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1. Introduction. A large part of the theory of Hardy spaces H^p on Euclidean space has been successfully extended to the setting of homogeneous Lie groups. This includes the duality of H^1 and BMO, the atomic decomposition, maximal function characterizations and boundedness of singular integral operators on H^p. However one aspect of the classical theory has until now been conspicuous by its absence: whereas $H^1(\mathsf{R}^n)$ was originally defined by Stein and Weiss [20] as the space of L^1 functions all of whose Riesz transforms are in L^1, not even on the simplest non-abelian nilpotent groups has any analogue of this characterization by singular integral operators been found. Attempts to approach the problem via subharmonicity on the Heisenberg group have failed.

To provide such a characterization is the goal of this paper. Our starting point is the recent breakthrough of A. Uchiyama [23], who completely determined which families of singular integral operators characterize $H^1(\mathsf{R}^n)$. The proof was constructive and almost purely real-variable in nature, making no use of subharmonicity. Using his techniques we shall show that for a broad class of groups G, the stratified groups, $H^1(G)$ may indeed be characterized by certain collections of singular integrals. The Heisenberg group H^n is studied in considerably greater detail. There we are able to describe all families having the desired property; the description is couched in terms of representation theory.

A substantial portion of our work consists of a relatively straightforward translation of Uchiyama's argument from Euclidean space to a general nilpotent group. However two new obstacles do arise. Both involve the group Fourier transform, which on nonabelian nilpotent groups is far less well understood than on Euclidean space. The first new hurdle encountered is a certain inversion problem which we now describe. By a singular integral operator will be meant a left-invariant convolution operator of the form $Kf(x) = \alpha f(x) + (f * k)(x)$, where α is a constant and k is a principal-value distribution on G as defined in Section 2. Given a finite collection $\{K_j\}_{j=1}^m$ of real singular integral operators, satisfying certain hypotheses, the problem is to show that given any unit vector $\nu \in \mathsf{R}^m$ there exist operators L_j such that

$$\begin{cases} \sum K_j^* L_j = I \\ \sum \nu_j L_j = 0. \end{cases} \tag{$*$}$$

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