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VOLUMES IN SEIFERT SPACE

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In [13], Thurston has described eight geometries which are relevant to the study of three-dimensional manifolds. Of these, hyperbolic geometry is, in his words, "by far the most interesting, the most complex, and the most useful." Naturally attached to hyperbolic space is the notion of Gromov volume—if M is a closed manifold, one attaches to the fundamental class of M a "simplicial volume" (as in [7]), which is defined topologically but which equals the volume of M as a hyperbolic manifold in the event that M has a hyperbolic structure.

If M has a geometric structure in the sense of [13], then the Gromov volume of M will be zero unless M is hyperbolic. Thus, Gromov volume detects the "hyperbolic part" of M.

The properties of Gromov volume have proved very fruitful in the study of hyperbolic 3-manifolds—see [6], [11] for a discussion.

In this paper, we observe a similar phenomenon for another of Thurston's geometries, which we call Seifert space geometry, and which is #6 on his list [13]. It is the geometry applicable to nontrivial Seifert fiber bundles, and from many points of view ranks a clear second among Thurston's geometries.

We attach to Seifert space a volume invariant analogous to Gromov volume, which is defined as a topological invariant of a compact, oriented 3-manifold M. We also provide, as in [1], some crude estimates for this invariant. In the event that M is a 3-manifold with a Seifert space structure, this invariant coincides with the volume of M in this geometry.

We also show that this Seifert volume shares many of the formal properties of Gromov volume. It follows from this that a 3-manifold with geometric structure and nonvanishing Seifert volume must have either hyperbolic or Seifert space structure, and that none of the other six geometries carry topological volumes with similar properties.

We carry out a complete analysis of Seifert volume for 3-manifolds with Seifert space structure. This is possible because the set of all such 3-manifolds has a fairly simple description. We then turn to the harder question of analyzing this invariant in the context of hyperbolic 3-manifolds, where we provide examples of hyperbolic 3-manifolds whose Seifert volume is zero, and examples of hyperbolic 3-manifolds whose Seifert volume is nonzero.

Our original interest in this invariant stems from our work [1] on real projective foliations. Indeed, the group $\widetilde{\mathrm{PSL}(2,\mathbb{R})}$ acts on Seifert space, and leaves invariant a codimension 1 real projective foliation. So if M has the

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